School of Physics and Astronomy

Junior Honours **Thermodynamics** GJA 2019-2020

Tutorial 8: Other systems

1 Questions: Thermodynamics in other systems

1. **EXAM-STYLE QUESTION: A rubber band**

   (a) Derive the so-called 'energy equation'

   $$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

   for a simple substance. If the equation of state of the substance is known, the energy equation allows one to calculate the dependence of the internal energy on volume.

   Hint: Use the Maxwell relation derived from the Helmholtz function.

   (b) Write down the analogous result for rubber band, where work done is tension times extension. Why is it different from force being the differential of energy?

   (c) The equation of state of a rubber band is

   $$\mathcal{F} = aT\left(\frac{L}{L_0} - \left(\frac{L_0}{L}\right)^2\right)$$

   where a is a constant and L_0 is the unstretched length. For this band, show that U is a function of T only.

   (d) If $L_0 = 1$ metre and $a = 1.3 \times 10^{-2}$ N K^{-1}, calculate the work done on the band and the heat rejected when it is stretched isothermally and reversibly from 1 m to 2 m at $T = 300$ K.

2. **An adiabatic variation on the rubber band.**

   If the rubber band in the previous problem is stretched *adiabatically* and reversibly from 1 m to 2 m, by how much does its temperature rise? (Take the heat capacity to be $C_L = 1.2$ J K^{-1}.)

3. **Entropy of diamond**

   The low temperature specific heat of diamond varies with temperature as:

   $$c_p = 124\left(\frac{T}{\theta_D}\right)^3 \quad \text{kJ kg}^{-1}\text{ K}^{-1}$$

   where the Debye temperature $\theta_D = 1860$ K. What is the entropy change of 1 g of diamond when it is heated at constant pressure from 4 K to 300 K?

   The Debye temperature of graphite is $\theta_D = 1500$ K, and

   $$c_p = 89\left(\frac{T}{\theta_D}\right)^3 \quad \text{kJ kg}^{-1}\text{ K}^{-1}$$

   Given that diamond has higher density than graphite, discuss whether the phase boundary on a PT phase diagram has positive or negative slope.? Does the equation for specific heat satisfy the Third Law, that the change in entropy in any process becomes zero at 0K?

   (The atomic weight of carbon is 12.)

4. **Planck's Law for frequencies**

   Taking care about units and definitions, use the relation $\nu = c/\lambda$, rewrite the Planck distribution

$$u_\lambda(\lambda, T) = \frac{hc^2}{\lambda^5}\left(\frac{1}{e^{hc/\lambda k_B T} - 1}\right)$$

   in terms of frequency:

$$u_\nu(\nu, T) = \frac{h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_B T) - 1}$$

5. **Planck's Law**

   The equation of state for radiation in a cavity is $P = U/3V$, where U(T) is the energy and V the volume. Use this with the Central Equation to show that the energy density varies as the fourth power of temperature.

   Starting from Planck's ansatz relation between u and s $\qquad \frac{\partial^2 s}{\partial u^2} = \frac{c_1(\lambda)}{u(u+c_2(\lambda))}$

   and integrating, derive the Planck distribution $u_\lambda(\lambda, T)$. You will need to use the Rayleigh and Wein limits to reintroduce λ via c_1 and c_2

   The total energy density must be the integral of the energy density over all frequencies.

   Show that the Planck distribution of energy between wavelengths satisfies the requirement that the total energy density varies as the fourth power of temperature. Show that the classical distribution ($u_\lambda(\lambda, T) = k_B T \lambda^{-4}$) with each mode having λ-independent energy $k_B T$ gives divergent energy.

   What would the Boltzmann distribution give?

6. **Photon Gas**

   (a) For a photon gas, the equation of state can be written $P = \frac{u(T)}{3}$ where $u(T)$ is the specific internal energy, which depends only on temperature. Show that the internal energy of a photon gas varies as the fourth power of temperature, $U = kVT^4$, where k is a constant.

   (b) An evacuated cylinder of volume $1m^3$ contains a photon gas confined by a piston. At what temperature would the cylinder need to be for the piston to move outward against atmospheric pressure (10^5Pa) ($k = 7.56 \times 10^{-16} J/m^3/K^4$)

   (c) Show that the Heat Capacity C_v of a photon gas obeys the Third Law, and evaluate the entropy of a photon gas, assuming that S=0 at T=0.

   (d) Show that the Gibbs Free Energy of a photon gas is zero and comment on the implication for creation and absorption of photons at a cavity wall.

   (e) Using the fact that $G = 0$, and *without* using the equation of state, prove that the pressure of a photon gas depends on temperature only.

7. **Temperature of the sun.**

   By assuming that both the earth and the sun are perfect black bodies that radiate equally in all directions use the Stefan Boltzmann law to estimate the temperature of the sun. (You will need Temperature of the earth $\approx 287K$, Radius of the sun $\approx 6.96 \times 10^8 m$, Distance from earth to sun (astronomical unit) $\approx 1.5 \times 10^{11} m$.

   Comment on the various assumptions you have made.

   Hint: Think of how much area the earth takes up relative to the total emission surface of the sun at the orbital distance. Assume there is a radiative equilibrium on the earth. Note the earth's radius cancels out during this calculation.

   Historical note: Stefan, lacking accurate knowledge of the relevant distances, performed this calculation by using a measure of the incident flux on earth. This was measured using huge reflecting mirrors and target objects!

8. **Black hole entropy**

A black hole has only three independent variables, mass, charge and angular momentum. When a particle falls into a black hole, information (entropy) is "lost".

It has been predicted that the entropy of a (non-rotating, uncharged) black hole depends on a single variable, such as its surface area (A) measured in Planck areas, $S = k_B A c^3/4G\hbar$. This is called the "Bekenstein-Hawking formula". Use it to calculate the temperature of a mini black hole of mass $M = 10^{12}$ kg, and a black hole the size of the sun $M = 2 \times 10^{30}$ kg.

Hint: You will need to identify, and find values for, k_B, G, $\hbar$ *and* c. *The radius of a black hole depends only on its mass* $r = 2Gm/c^2$ *(Schwartzchild radius), and its energy also depends only on the mass* $U = mc^2$ *The Central Equation of Thermodynamics lets you introduce* T *explicitly. The same equation suggests that the appropriate variables are* U *and* V*, so that* **formally** $S = S(U, V)$. *Assume that* S *is a function of* U *only.*