School of Physics & Astronomy

Thermal Physics

PHYS09061 (SCQF Level 9)

Thursday 7th May, 2015 09:30 – 12:30
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **ALL** questions from Section A and **THREE** questions from Section B & C, answering at least one question from each section.

Special Instructions

- Only the supplied Electronic Calculators may be used during this examination.
- A sheet of physical constants is supplied for use in this examination.
- Attach supplied anonymous bar codes to *each* script book.

Special Items

- School supplied calculators
- School supplied Constant Sheets
- School supplied barcodes

Chairman of Examiners: Prof A Trew
External Examiner: Prof S J Clark

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

Section A: Answer ALL the questions from this Section

A.1 For a magnetic system at zero pressure, the internal energy is $U = HM - TS$ where H is the applied field and M is the magnetisation.

Use $dU = TdS + HdM$, the magnetic equivalent of the central equation, to prove that

$$\left(\frac{\partial T}{\partial M}\right)_S = \left(\frac{\partial H}{\partial S}\right)_M$$

explaining any assumptions that you make. **[5]**

A.2 Write down the mathematical expression defining thermal expansivity at constant pressure **[1]**

One statement of the Third Law of Thermodynamics is that the change in entropy in any isothermal process tends to zero as T tends to zero. Write this statement mathematically in terms of a partial differential, and use it to show that the thermal expansivity of any material is zero at absolute zero. **[4]**

A.3 Define the coefficient of performance (efficiency) for a refrigerator in terms of work input, W, and heat extracted from the cold reservoir, Q_c. **[1]**

A company markets a device which can extract heat at 0.4kW from a cold compartment of a fridge *and* deliver 1kW heating to the surrounding room. It is claimed that the device uses the same amount of electricity as a 100W lightbulb. Is the claim thermodynamically plausible? Explain your answer stating any assumptions made. **[4]**

A.4 Consider integers from 1 to 9.

a) What is the number of all possible 4-integer strings? **[1]**

b) What is the number of 4-integer strings with two integers being the same and the other two being different? **[3]**

c) For a string of six integers, three of which are the same and the rest are all different, what is the number of unique strings obtained by permutations of the original one? **[1]**

A.5 A random variable x can take values from the range $[0, \infty)$. The probability of a measured value of x to be in the range $(x, x + dx)$ is given by $p(x)dx$.

a) Write down the normalisation condition for $p(x)$ and the expression for the average value of x. **[1]**

b) Assume $p(x) = Ae^{-x+1}$, where A is a constant. Find A and the average value of x. **[4]**

A.6 Consider an ideal gas of N non-polar molecules at a temperature T. The molecules can be either inside or outside a capacitor. The energy of each molecule inside the capacitor is given by $\epsilon = -\frac{1}{2}\alpha\mathbf{E}^2$, where α is the polarisability of the molecules and $\mathbf{E}$ is a constant electric field inside the capacitor. Outside the capacitor molecules have zero energy. Calculate the ratio of the number of molecules inside and outside the capacitor. **[5]**

Section B: Answer AT LEAST ONE of the questions in this Section

B.1 This question considers the thermodynamics of black body cavity radiation.

a) Starting from the central equation, prove that

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P.$$

[3]

b) For a photon gas, the equation of state can be written $P = \frac{u(T)}{3}$ where $u(T)$ is the specific internal energy, which depends only on temperature. Show that the internal energy of a photon gas varies as the fourth power of temperature, $U = kVT^4$, where k is a constant. **[4]**

c) Explain why u depends only on temperature, but U has a volume dependence, where u and U have their standard meanings. **[1]**

d) An cylinder of volume 1m^3 contains a photon gas confined by a piston. At what temperature would the cylinder need to be for the piston to move outward against atmospheric pressure, 10^5Pa? ($k = 7.56 \times 10^{-16}$ $\text{Jm}^{-3}/\text{K}^{-4}$) **[4]**

e) Show that the heat capacity C_v of a photon gas obeys the Third Law, and evaluate the entropy of a photon gas, assuming that $S = 0$ at $T = 0$. **[4]**

f) Calculate the Gibbs Free Energy of a photon gas and comment on the implication for creation and absorption of photons at a cavity wall. **[4]**

B.2 This question examines the adiabatic expansion of an ideal gas.

a) Write down the Central Equation of Thermodynamics. **[2]**

The heat capacities at constant volume and pressure are *defined* by:

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_V ; \qquad C_P = \left(\frac{\partial Q}{\partial T}\right)_P$$

where Q is the heat supplied to the material and T is the temperature of the material.

b) By considering constant volume heating, show that the internal energy of an ideal gas is:

$$U = C_V T.$$

You may assume that $U(T = 0) = 0$. **[2]**

c) Show that for an adiabatic process: **[2]**

$$C_V dT = -PdV,$$

and derive a similar expression for C_P.

d) Using the previous results, show that for a reversible adiabatic process involving an ideal gas:

$$\int \frac{dT}{RT} = -\int \frac{dV}{c_V V} = \int \frac{dP}{c_P P}.$$

(Notice that this equation involves *specific* heat capacities.) **[2]**

e) Hence prove that an adiabatic expansion of an ideal gas obeys the relationship:

$$PV^\gamma = constant,$$

and find an expression for γ in terms of the heat capacities. **[4]**

In a Joule expansion process a gas expands adiabatically into a vacuum from an initial volume V_1 to final volume V_2.

f) Explain why entropy is not conserved in this process. **[1]**

g) Show that in general the temperature change is

$$\Delta T = \frac{V_2 - V_1}{C_V}\left[P - T\left(\frac{\partial P}{\partial T}\right)_V\right].$$

[4]

h) Evaluate the temperature change for an ideal gas starting at 300K when the volume is doubled in an adiabatic expansion. **[3]**

B.3 This question considers the thermodynamics of a heat engine. Assume throughout that there is no reduction in efficiency due to friction, mechanical imperfections etc.

a) Write down the Carnot efficiency of a heat engine running between two reservoirs at temperatures T_2 and T_1, where $T_2 > T_1$. **[2]**

A heat engine cycle comprises two isotherms ($T_2 > T_1$) and two isochores ($V_2 > V_1$).

b) Draw the cycle on a PV diagram, and again on a TS diagram, labelling the processes and end states. Identify where heat enters and leaves the system, and where work is done. **[4]**

c) Considering an ideal gas as the working substance, evaluate the heat input and work done in each process. **[4]**

d) Hence evaluate the efficiency of the engine, and show that for very large expansions it becomes equal to the Carnot efficiency. **[6]**

e) If the engine was scaled up to double size, how would this affect the efficiency? **[2]**

f) Explain why using an isothermal process in a heat engine leads to low power output. **[2]**

Section C: Answer AT LEAST ONE of the questions in this Section

C.1 In Pauli's model of paramagnetism, N electrons are considered to form an ideal Fermi-gas in the presence of a magnetic field $\mathbf{H}$. The average numbers of electrons with spins aligned with the field, $\bar{n}_+$, and against the field, $\bar{n}_-$, and occupying a quantum state with energy ϵ are given by the corresponding Fermi-Dirac distributions

$$\bar{n}_+(\epsilon) = \frac{1}{\exp\left[\frac{\epsilon-\mu+\mu_B H}{k_B T}\right]+1},$$
$$\bar{n}_-(\epsilon) = \frac{1}{\exp\left[\frac{\epsilon-\mu-\mu_B H}{k_B T}\right]+1},$$

where μ is the chemical potential, μ_B is the Bohr magneton, k_B is Boltzmann's constant, and H is the magnetic field strength. The density of states is the same for electrons with their spins parallel or antiparallel to $\mathbf{H}$, and is given by

$$g(\epsilon) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} V\sqrt{\epsilon}.$$

Here, m is the mass of an electron, V is the volume of the gas, and $\hbar$ is the Planck constant.

Assume $T = 0$ and consider only linear order in $\mu_B H \ll \mu$.

a) Sketch $\bar{n}_+(\epsilon)$ and $\bar{n}_-(\epsilon)$. **[2]**

b) Write down the condition relating the chemical potential μ to the total number of electrons N. Determine the value of Fermi energy $\epsilon_F \equiv \mu$. **[6]**

c) Calculate the total energy of the gas. **[5]**

d) To linear order in H, calculate the average magnetisation of the system per unit volume

$$M = -\mu_B \frac{N_+ - N_-}{V},$$

where N_+ and N_- are the total number of electrons with spins aligned parallel or antiparallel to the magnetic field. **[5]**

e) Show that the magnetic susceptibility χ is given by

$$\chi = \frac{3}{2}\frac{N}{V}\frac{\mu_B^2}{\epsilon_F}.$$ **[2]**

C.2 Consider a system of N non-interacting quantum rotators at temperature T. The energy levels of a single rotator are quantised by the orbital quantum number l and are given by

$$E_l = \frac{l(l+1)\hbar^2}{2I},$$

where I is the moment of inertia of the rotator, $\hbar$ is Planck's constant, and l takes integer values from zero to infinity. Each energy level is $(2l+1)$-degenerate. Assume that the temperature is low.

a) Write down the expression for the partition function Z in the semi-classical limit in terms of the sum over possible quantum numbers. Since the temperature is low, approximate Z by the first non-trivial term in the sum. **[5]**

b) Discuss the range of temperatures where this is a good approximation. **[3]**

c) Calculate the mean energy $\bar{E}$, the free energy F, and the entropy S of the system. **[7]**

d) State the conditions when the semi-classical entropy becomes unphysical and explain this result. **[5]**

C.3 A molecule of mass m is in contact with a thermal bath at temperature T. It performs a random walk in a gravitational field of strength g acting downwards. At each step of the random walk the molecule can go up, down or stay in the same place. Correspondingly, its energy increases by mgl, decreases by the same amount, or does not change; here l is the length of a single step. The molecule has taken N steps.

a) Calculate the canonical partition function $Z(1)$ for a single step. **[4]**

b) What is the probability for a step to be upwards? Downwards? **[2]**

c) Write the canonical partition function for the whole random walk. **[2]**

d) Calculate the average total distance L moved along the vertical direction. **[6]**

e) Derive expressions for L in the limits of $mgl \ll k_B T$ and $mgl \gg k_B T$, where k_B is Boltzmann's constant, and explain the results. **[6]**