School of Physics & Astronomy

Thermal Physics

PHYS09061 (SCQF Level 9)

Friday 6$^{\text{th}}$ May, 2016 09:30 − 12:30
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **ALL** questions from Section A and
THREE questions from Section B & C, answering at least
one question from each section.

Special Instructions

- Only authorised Electronic Calculators may be used during this examination.
- A sheet of physical constants is supplied for use in this examination.
- Attach supplied anonymous bar codes to *each* script book.

Special Items

- School supplied Constant Sheets
- School supplied barcodes

Chairman of Examiners: Prof A Trew
External Examiner: Prof S J Clark

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS
EXAMINATION.

Section A: Answer ALL the questions from this Section

A.1 Define the coefficient of performance (efficiency) for a heat pump in terms of work input, W, and heat delivered, Q_h. [1]

What is the maximum (Carnot) coefficient of performance, η_c of a heat pump operating between heat reservoirs at T_1 and T_2? [2]

Mathematically, it is possible for $\eta_c < 0$: explain what that means physically in terms of the Second Law? [2]

A.2 State the three versions of the Second Law of Thermodynamics, according to Kelvin, Clausius and Carnot's theorem. [5]

A.3 The Gay-Lussac Law states that "The pressure of a gas of fixed mass and fixed volume is directly proportional to the gas's absolute temperature". Write this sentence as a partial differential equation. [1]

Is the Gay-Lussac Law true or false for the van der Waals gas? If true, find the proportionality constant; if false, find the lowest order nonlinear term. [4]

A.4 a) There are 24 students in a class, none of them born in a leap year. Find the probability that at least two students have their birthdays on the same day. [2]

b) What is the probability that all four aces are next to each other in a well-shuffled deck of 52 cards? [3]

A.5 a) Give the definitions of a microstate and a macrostate. [2]

b) Write down Boltzmann's law for the entropy, explaining the meaning of the symbols in the formula. [3]

A.6 An atom can be in either of two possible states with energies nk_BT, where $n = 1$ or 2. Consider two such atoms that interact with each other at a temperature T. The energy of their interaction is given by $k_BT(n-m)^2$, where n and m label the states of the atoms. How much more probable is the configuration with $n = m = 1$ than the configuration with $n = 1$ and $m = 2$? [5]

Section B: Answer AT LEAST ONE of the questions in this Section

B.1 This question examines reversible and irreversible processes.

(a) What is the relationship between the chemical potential and the Gibbs free energy for a single component system? **[2]**

(b) A system comprising 1 kg of ice in thermal contact with the environment at 0^oC melts to become water at 0^oC. The surrounding environment remains infinitesimally above 0^oC throughout.

    i) How much heat is supplied? **[1]**

    ii) By how much does the entropy of the system and the surroundings change. **[4]**

    iii) Discuss whether the process is reversible. **[3]**

(c) An isolated system initially containing 6 kg of water at 300K and 4 kg of ice at 273K.

    i) After a long period of time, how much of the ice will have melted and why is this different from part (b)? **[2]**

    ii) Sketch how the chemical potentials of the ice and water vary with time. Numerical values are not required, but you must label the axes and curves. **[4]**

    iii) What is the entropy change of the system? **[4]**

Latent heat of melting: 334kJ/kg; Heat capacity of water: 4.2kJ/kg/K

B.2 This question examines the liquid-vapour transition for water, which has a critical point at 647K, 22 MPa, and a triple point at 273K, 600Pa, where the latent heat of vapourisation is 2.3×10^6 J/kg.

(a) Explain what is meant by the *critical point* of a fluid. What is the latent heat of vapourisation for water at the critical point? **[4]**

(b) The Clausius-Clapeyron equation can be written:

$$\left(\frac{dP}{dT} \right) = \frac{l}{T(v_2 - v_1)}.$$

Define all the terms, and name their SI units. **[4]**

(c) A schoolboy claims that the Clausius-Clapeyron slope is the ratio of bulk modulus (K) to the thermal expansion (β), because...

$$K/\beta = V \frac{dP}{dV} \Big/ \frac{1}{V} \frac{dV}{dT} = \frac{dP}{dV} \frac{dV}{dT} = \frac{dP}{dT}.$$

Explain the schoolboy's errors. **[4]**

(d) For boiling, the Clausius-Clapeyron equation can be reduced to

$$\left(\frac{dP}{dT} \right) = \frac{Pl}{nRT^2}.$$

Derive this equation, pointing out the *two* approximations which are required. **[4]**

(e) Assuming that the latent heat varies linearly with temperature between the triple and critical points, show that the equation for the liquid-gas coexistence curve has the form

$$P = AT^\alpha \exp(B/T),$$

and sketch the PT phase diagram. You need not determine the constants A, B and α. **[4]**

B.3 This question examines an idealised engine.

(a) The ideal gas Otto cycle is an idealisation of the internal combustion engine, which comprises two adiabats and two isochores. Show the cycle on a PV diagram, explaining which segments represent doing useful work, fuel compression, burning, and exhaust. **[4]**

(b) Identify the processes during which heat and work flow in or out of the system. Give expressions for these heat and work flows in terms of an ideal gas with heat capacities C_v and C_p. **[4]**

(c) Give an expression for the net work output per cycle. **[2]**

(d) Show that for an appropriate definition of V_1 and V_2, the efficiency of the Otto cycle is **[6]**
$$1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1}.$$

You may use without proof the fact that $TV^{\gamma-1}$ is constant along an adiabat.

(e) How would the efficiency change if the combustion occurred in an oxygen-CO_2 mixture, rather than air? Assume 20% O_2 composition in both cases. **[4]**

Section C: Answer AT LEAST ONE of the questions in this Section

C.1 An ideal gas of N ultrarelativistic particles is confined to a volume V. The gas is in thermal equilibrium with a heat bath at temperature T. The energy of each particle with linear momentum $\boldsymbol{p}$ is given by $\epsilon(\boldsymbol{p}) = cp$, where $p = |\boldsymbol{p}|$ and c is the speed of light in vacuum.

a) Derive the Maxwell-Boltzmann distribution for the probability of a single particle to have its momentum between p and $p + dp$. What is the most probable value of p? [4]

b) Calculate the single-particle partition function $Z(1)$. [4]

c) Employ the semi-classical approximation to show that the partition function of the gas is given by

$$Z = \frac{V^N}{\pi^{2N} N!} \left(\frac{k_B T}{\hbar c} \right)^{3N},$$

where $\hbar$ and k_B are the reduced Planck and Boltzmann constants, respectively. Explain the origin of the $1/N!$ factor. [2]

d) Calculate the average energy and entropy of the gas. Use Stirling's approximation where necessary. Calculate the heat capacity of the gas and compare it to the Dulong-Petit law for a monatomic non-relativistic ideal gas. [5]

e) Using the expression for the entropy derived in part d), discuss when the semi-classical approximation becomes unphysical. [5]

C.2 In this question you will calculate the heat capacity of an ideal solid within the Einstein and Debye models. The solid consists of N atoms that occupy volume V in thermal equilibrium with a heat bath at temperature T.

a) Use the equipartition principle to estimate the heat capacity of the solid. You may assume that the atoms perform classical harmonic oscillations around their equilibrium lattice positions. [3]

b) Consider a quantum oscillator with the linear frequency ν. Its energy is quantised according to

$$\epsilon_n = h\nu\left(n + \frac{1}{2}\right), \qquad n = 0, 1, \ldots \infty,$$

where h is the Planck constant. Calculate the partition function for the oscillator. [5]

c) The ideal solid can be thought of as a collection of independent normal modes of lattice vibrations, each being a quantum oscillator. Using the expression for the single-oscillator partition function, show that the average energy of the solid is given by

$$\bar{E} = \int_0^\infty \left[\frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} + \frac{1}{2}\right] h\,\nu\,g(\nu)d\nu,$$

where $g(\nu)$ is the density of states and k_B is the Boltzmann constant. [6]

d) In the Einstein model of ideal solids,

$$g(\nu) = 3N\delta(\nu - \nu_E),$$

where $\delta(x)$ is the Dirac delta-function, while the Debye model assumes

$$g(\nu) = \begin{cases} \frac{9N}{\nu_D^3}\nu^2, & \nu \leq \nu_D \\ 0, & \nu > \nu_D \end{cases}$$

where ν_E and ν_D are constants. Compare the heat capacity predicted by the two models for very high and very low temperatures. [6]

C.3 The conducting electrons in a metal can be modelled as a gas of fermions of spin $1/2$ and density $n = N/V$, where N is the number of electrons and V is the volume. Assume that $T = 0$.

a) Describe the ground state of non-interacting electrons. Define and calculate the Fermi energy ϵ_F for this system. You may assume that the density of states is given by

$$g(\epsilon) = \frac{V\sqrt{\epsilon}}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2},$$

where ϵ is the energy, m is the mass of the electron, and $\hbar$ is the reduced Planck constant.

[5]

b) Calculate the energy of the ground state of the system

$$\int_0^{\epsilon_F} \epsilon\, g(\epsilon)d\epsilon.$$

Show that it can be written as

$$\frac{3}{5}\frac{\hbar^2}{2m}V\left(6\pi^2\right)^{2/3}\left[n_+^{5/3} + n_-^{5/3}\right],$$

where $n_+ = N_+/V$ and $n_- = N_-/V$ are the densities of spins up and spins down, respectively, and in the ground state $N_+ = N_- = N/2$.

[5]

c) The Coulomb interaction between the electrons can be described by an effective contribution to the energy of the system in the form

$$\alpha n_+ n_-,$$

where $\alpha > 0$ is the interaction strength. Imagine a small perturbation to the ground state of the system, in which there are slightly more spins up than down, $n_+ = n/2 + \delta$ and $n_- = n/2 - \delta$, with $\delta > 0$. Evaluate the energy of this state in the presence of interaction between the electrons to second order in δ/n.

[5]

d) Determine the strength of the effective interaction α for which such a perturbation lowers the energy of the system. What happens to the magnetisation of the metal at this point?

[5]

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