School of Physics & Astronomy

Thermal Physics

PHYS09061   (SCQF Level 9)

Friday 12$^{\text{th}}$ May, 2017   09:30-12:30
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

> Answer **ALL** questions from Section A and
> **THREE** questions from Section B & C, answering at least
> one question from each section.

Special Instructions

- Only authorised Electronic Calculators may be used during this examination.
- A sheet of physical constants is supplied for use in this examination.
- Attach supplied anonymous bar codes to *each* script book.

Special Items

- School supplied Constant Sheets
- School supplied barcodes

Chairman of Examiners: Prof A Trew
External Examiner: Prof S J Clark

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

Section A: Answer ALL the questions from this Section

Throughout this paper, $\beta = 1/k_B T$ where k_B is Boltzmann's constant and T is temperature.

A.1 Write down the Clausius Inequality for a system undergoing a cyclic process, exchanging heat with a number of reservoirs, and show how this implies the existence of a state variable, the entropy. Define all your terms carefully. **[5]**

A.2 For a magnetic system at zero pressure, the internal energy is $U = HM - TS$ where H is the applied field and M is the magnetisation. The equivalent central equation is $dU = TdS + HdM$.

Prove that

$$\left(\frac{\partial T}{\partial M}\right)_S = \left(\frac{\partial H}{\partial S}\right)_M$$

explaining any assumptions that you make. **[5]**

A.3 (a) Starting from the Central Equation $dU = TdS - PdV$, show that

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P\,.$$

You may use a Maxwell relation without proving it. **[3]**

(b) For a photon gas, the equation of state can be written

$$P = \frac{u(T)}{3}$$

where $u(T)$ is the internal energy per unit volume which depends only on temperature, and $u(0) = 0$. Show that $u = \kappa T^4$, where κ is a constant. **[2]**

A.4 (a) In statistical mechanics, explain what is meant by a microstate and a macrostate. **[2]**

(b) State the Boltzmann law for the entropy of a macrostate, explaining the meaning of all the symbols that you use. **[3]**

A.5 A particle can occupy one of a set of non-degenerate energy levels $\{\epsilon_1, \epsilon_2, \epsilon_3, \ldots\}$ and is in contact with a heat reservoir at temperature T.

(a) What is the canonical partition function $Z(1)$ for this particle? [1]

(b) Show that the mean energy of this particle can be written as

$$\bar{\epsilon} = -\frac{\mathrm{d}}{\mathrm{d}\beta} \ln Z(1) \ .$$

[4]

A.6 The Fermi-Dirac distribution is given by

$$\bar{n}(\epsilon) = \frac{1}{\mathrm{e}^{\beta(\epsilon-\mu)} + 1}$$

where $\bar{n}(\epsilon)$ is the expected number of particles in energy level ϵ at chemical potential μ.

Sketch the form of this distribution in the limit $T \to 0$ and explain why it is implied by the Pauli Exclusion Principle. [5]

Section B: Answer AT LEAST ONE of the questions in this Section

B.1 The question examines your ability to deal with a thermodynamic quantity you haven't seen before, the "molar specific non-ideality". You can assume the van der Waals equation of state $RT = (P + a/v^2)(v - b)$.

The "molar specific non-ideality" of a fluid is defined by $y = Pv - RT$

(a) Explain why y is a state variable. **[2]**

(b) What is the value of y for an ideal gas? **[2]**

(c) Derive expressions for $y(P, v)$ and $\left(\frac{\partial y}{\partial T}\right)_v$ for a van der Waals fluid. **[6]**

(d) A van der Waals fluid undergoes an expansion process at constant y. Calculate the associated cooling coefficient $\left(\frac{\partial T}{\partial v}\right)_y$. **[4]**

(e) The mathematical expression for the cooling coefficient in (d) should become infinite for $b = 0$, but obviously the material cannot undergo an infinite amount of cooling. With reference to the expression for y, explain what causes this apparent contradiction. **[3]**

(f) The y-inversion curve is defined as the line in (T, V) space for which the cooling coefficient is zero. Draw this curve, indicating relevant values, and explaining which regions of the graph are physically reasonable. **[3]**

B.2 This question considers a body of liquid at temperature T with constant heat capacity C_P. It is placed in thermal contact with its environment which is at temperature T_0 and assumed large enough to have constant temperature. Assume that the equilibration process is isobaric, but cater for the possibility that the system is either cooling or warming.

(a) Starting from the central equation, $dU = TdS - PdV$, show that the entropy change in a system with constant heat capacity, C_P, can always be written

$$dS = \frac{C_P}{T}dT - \frac{V}{T}dP$$

and write down the simplification of this expression for isobaric process. **[3]**

(b) What are the SI units of C_P? **[1]**

(c) Calculate the entropy change of the system and of the surroundings. **[6]**

(d) Show that the entropy change of the universe due to this process can be written in the form:
$$\Delta S = C_P f(x) \,,$$
where $f(x) = x - \ln(1 + x)$, and determine how x depends on T and T_0. **[4]**

(e) Sketch $f(x)$, and with reference to the sketch, explain what features of the function $f(x)$ relate to

(i) the Second Law; and

(ii) the Third Law. **[7]**

B.3 This question considers the thermodynamics of a heat engine.

(a) Write down the Carnot efficiency of a heat engine running between two reservoirs at temperatures T_2 and T_1, where $T_2 > T_1$. **[2]**

A heat engine cycle comprises two isotherms ($T_2 > T_1$) and two isochores ($V_2 > V_1$).

(b) Draw the cycle on a PV diagram, and again on a TS diagram, labelling the processes and end states, and identifying where heat enters and leaves the system, and where work is done. **[4]**

(c) Considering an ideal gas as the working substance, evaluate the heat input and work done in each process. **[4]**

(d) Hence evaluate the efficiency of the engine, and show that for very large expansions it becomes equal to the Carnot efficiency. **[6]**

(e) If the engine was scaled up to double size, how would this affect the efficiency? **[2]**

(f) Explain why using an isothermal process in a heat engine leads to low power output.

[2]

Section C: Answer AT LEAST ONE of the questions in this Section

C.1 A triangular lattice comprises N regular positions and $2N$ interstitial positions as shown in the figure.
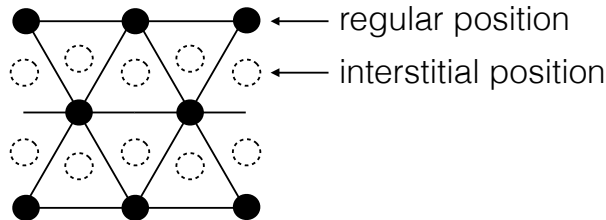
N atoms are arranged on the lattice in such a way that n of them are at interstitial positions. The potential energy of an atom at an interstitial position is an amount ϵ higher than an atom at a regular position. The atoms interact weakly with each other, and the entire system is in contact with a heat bath at temperature T.

(a) Show that the number of configurations with n atoms at interstitial positions is

$$\Omega(n) = \binom{N}{n}\binom{2N}{n} .$$

[3]

(b) Using Stirling's approximation, show that the entropy of the macrostate n can be written as

$$S(n) \approx Nk_B \left[c - 2x \ln x - (1 - x)\ln(1 - x) - (2 - x)\ln(2 - x) \right]$$

where $x = n/N$, and both n and N are considered to be large. State the value of the constant c.

[6]

(c) By minimising the Helmholtz free energy for the system, show that the equilibrium value of x is obtained by solving a quadratic equation and given by

$$\bar{x} = \frac{\sqrt{8\mathrm{e}^{\beta\epsilon} + 1} - 3}{2(\mathrm{e}^{\beta\epsilon} - 1)} .$$

[7]

(d) Find the value of $\bar{x}$ in the $T \to \infty$ limit. Comment on the physical reason for the value you obtain.

[4]

C.2 A diatomic molecule has translational, rotational and vibrational degrees of freedom. The vibrational component can be modelled as a one-dimensional quantum harmonic oscillator with frequency ω. The energy levels of this system are

$$\epsilon_n = \hbar\omega \left(n + \frac{1}{2} \right)$$

where $n = 0, 1, 2, \ldots$.

(a) Show that the canonical partition function for the vibrational degrees of freedom is

$$Z(1) = \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \ .$$

[4]

(b) Obtain expressions for the mean and the variance of the energy in the vibrational degrees of freedom as a function of temperature. [9]

(c) Hence, or otherwise, identify a characteristic temperature T_{vib} at which the molecule is likely to be in an excited vibrational state. [3]

The energy levels for the rotational states are

$$\epsilon_\ell = \frac{\hbar^2 \ell(\ell+1)}{2I}$$

where the angular momentum state $\ell = 0, 1, 2, \ldots$ has degeneracy $2\ell + 1$, and I is the moment of inertia of the molecule.

(d) For an oxygen molecule, $\omega = 2.98 \times 10^{14}\,\text{Hz}$ and $I = 1.93 \times 10^{-46}\,\text{kg}\,\text{m}^2$. Determine the vibrational and rotational states that are typically occupied at room temperature. [4]

C.3 The energy levels of a spin 0 boson in two dimensions are

$$\epsilon_{n_x, n_y} = \frac{\hbar^2 \pi^2}{2mA}(n_x^2 + n_y^2)$$

where n_x and n_y are both positive integers, m is the mass of the boson and A is the area of the container.

(a) Show that the density of states for a single particle is

$$g(\epsilon) = \frac{Am}{2\pi \hbar^2} \ .$$

[7]

(b) Show that the mean number of bosons in a system of bosons in contact with a heat bath at temperature T and a particle reservoir at chemical potential μ is given by

$$\bar{N} = K \int_0^\infty d\epsilon \frac{1}{e^{\beta(\epsilon - \mu)} - 1} \ ,$$

stating the value of the constant K.

[3]

(c) Explain why we must have $\mu < 0$.

[3]

(d) Show that $\bar{N}$ diverges as $\mu \to 0$ from below.

[4]

(e) Use this result to determine whether this system of particles undergoes Bose-Einstein condensation.

[3]