School of Physics & Astronomy

THE UNIVERSITY *of* EDINBURGH

Thermal Physics

PHYS09061 (SCQF Level 9)

Monday 7th May, 2018 14:30 - 17:30
(May Diet)

Please read full instructions before commencing writing.

Examination Paper Information

Answer **ALL** questions from Section A and **THREE** questions from Section B & C, answering at least one question from each section.

Special Instructions

- Only authorised Electronic Calculators may be used during this examination.
- A sheet of physical constants is supplied for use in this examination.
- Attach supplied anonymous bar codes to *each* script book.

Special Items

- School supplied Constant Sheets
- School supplied barcodes

Chairman of Examiners: Prof A Trew
External Examiner: Prof S J Clark

ANONYMITY OF THE CANDIDATE WILL BE MAINTAINED DURING THE MARKING OF THIS EXAMINATION.

Section A: Answer ALL the questions from this Section

A.1 The low-temperature specific heat of diamond varies with temperature, T (in Kelvin) as

$$c_p = 124 \left(\frac{T}{1860}\right)^3 \text{ J/g/K} .$$

(a) Calculate the entropy change when 1g of diamond is heated at constant pressure from 4K to 300K. **[3]**

(b) Does this equation for specific heat satisfy the third law of thermodynamics? Justify your answer. **[2]**

A.2 By considering entropy as a function of T and V, show that for an ideal gas

$$\left(\frac{\partial S}{\partial T}\right)_P = \frac{nR}{T} + \left(\frac{\partial S}{\partial T}\right)_V .$$ **[5]**

A.3 The specific Helmholtz free energy of a particular gas is

$$f = cT^2 - \frac{a}{v} - RT\ln(v-b)$$

where a, b and c are constants. Calculate the pressure of the gas in terms of T and v. **[5]**

A.4 In statistical mechanics, the entropy $S(E)$ of a macrostate with energy E is defined as $S(E) = k_B \ln \Omega(E)$, where $\Omega(E)$ is the weight function.

(a) Explain what is meant by "the weight function". **[2]**

(b) By appealing to the principle of equal *a priori* probabilities, explain why entropy tends to increase over time. **[3]**

A.5 An electron is confined to a one-dimensional channel of length 6×10^{-12} m which is in contact with a heat reservoir at temperature T. The electron's energy is determined by its energy level $n = 1, 2, 3, \ldots$ through the formula $\epsilon_n = 0.01n^2$ eV. Determine the temperature T at which the electron is 20 times more likely to be in the ground state than the $n = 3$ state. **[5]**

A.6 (a) State which of the following quantities is fixed and which can fluctuate in the grand-canonical ensemble:

(i) Energy, E

(ii) Temperature, T

(iii) Particle number, N

(iv) Chemical potential, μ. [**2**]

(b) Write down the form of the grand-canonical distribution function in terms of the quantities (i)–(iv) above. [**3**]

Section B: Answer AT LEAST ONE of the questions in this Section

B.1 This question concerns a three-stage engine.

A heat engine cycle using one mole of ideal gas consists of the following three stages:

1. an isobaric compression from volume V_a to V_b at pressure P_a
2. an increase in pressure from P_a to P_b at a constant volume V_b
3. an adiabatic expansion from (P_b, V_b) to the original state at (P_a, V_a)

You may assume throughout that PV^γ is constant in the adiabatic process.

(a) Write down the relationship between heat capacity (C_v) and internal energy for an ideal gas. **[1]**

(b) Sketch this cycle on a PV plot. **[4]**

(c) At what point is the temperature highest, and what is its value? **[2]**

(d) Calculate the heat and work at each stage of the cycle. **[6]**

(e) Define the efficiency of the engine in terms of the heat input and output. **[3]**

(f) Show that the efficiency of the cycle described above is

$$\eta = 1 - \gamma \frac{r-1}{(r^\gamma - 1)}$$

where $r = V_a/V_b$. **[4]**

B.2 This question examines CO_2 going from solution to atmosphere.

(a) Write down the general expression relating chemical potential to Gibbs free energy. **[2]**

(b) Calculate the change in chemical potential of an ideal gas when pressurised isothermally from 1atm to 5atm. **[3]**

A 1.05 litre bottle contains 1.00 litres of fizzy lemonade pressurized to $P_g = 5$atm with CO_2 gas. The CO_2 concentration in the lemonade is initially $x = 0.2$ moles/litre. Treat the gaseous CO_2 as an ideal gas at 300K, and take the chemical potential of the dissolved CO_2 proportional to the concentration, $\mu = Ax$.

(c) What three thermodynamic quantities have equal value in the lemonade and in the gas above it? **[3]**

The bottle cap is briefly loosened, so that the gas comes fully into equilibrium with the atmosphere, but all the dissolved CO_2 remains in solution. The bottle is then sealed.

(d) What are the chemical potentials of the CO_2 in the lemonade, and in the gas above it, at the moment when the bottle is sealed? **[4]**

Now the bottle is shaken, which allows the gas and solution to come into equilibrium.

(e) Describe what happens and estimate how many moles of CO_2 come out of solution. **[5]**

(Hint: Because the volume of liquid is larger than the gas, assume that it acts as a reservoir of constant chemical potential.)

(f) Approximately how many times can the process of loosening and tightening the cap be repeated before the drink goes flat? **[3]**

B.3 This question concerns compressing an elastic solid.

(a) Show that for a piston of area A, moving some distance $\mathrm{d}\mathbf{x}$, subject to force $\mathbf{F}$, the two expressions for work, $\mathbf{F}.\mathrm{d}\mathbf{x}$ and $P\mathrm{d}V$ are equivalent. **[2]**

(b) Write down the central equation of thermodynamics. **[2]**

(c) Use this equation to show that

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P\ .$$ **[3]**

A block of solid material has an equation of state that involves two constants a and V_0:

$$P = aT\left(\frac{V_0^2}{V^2} - \frac{V}{V_0}\right)\ .$$

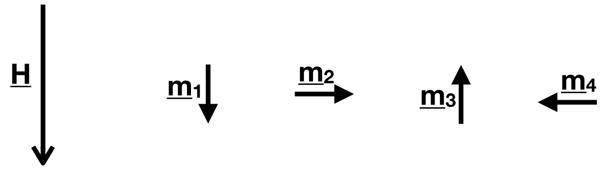
(d) Calculate the volume of the block at zero pressure. **[2]**

(e) Define and evaluate the isothermal bulk modulus of the material at $V = V_0$. **[5]**

(f) In a high pressure experiment, the block is compressed isothermally from V_0 to $V_0/2$. Calculate work done, change in internal energy, and heat supplied in the experiment. **[6]**

Section C: Answer AT LEAST ONE of the questions in this Section

C.1 This question concerns an assembly of N weakly-interacting dipoles, each of which can be oriented in one of four directions, $\underline{m}_1, \underline{m}_2, \underline{m}_3$ and $\underline{m}_4$, as shown in the figure. The magnitude of the dipole moment, m, is the same in all four configurations.



The energy of a dipole is given by $\epsilon(\underline{m}) = -\underline{m} \cdot \underline{H}$ where $\underline{H}$ is the applied field. The assembly is in equilibrium with a heat reservoir at temperature T.

(a) Determine the distinct energies that a single dipole ($N = 1$) can have, and the number of microstates corresponding to each one. **[3]**

(b) Show that the partition function for a single dipole, $Z(1)$, is

$$Z(1) = 2[1 + \cosh(\beta m H)]$$

where m and H are the magnitudes of $\underline{m}$ and $\underline{H}$, respectively. **[3]**

(c) Show that the mean energy of a single dipole is

$$\bar{\epsilon} = -mH\frac{\sinh(\beta m H)}{1 + \cosh(\beta m H)} \ .$$ **[3]**

The magnetisation of a system of N dipoles is defined as $M = m(N_\downarrow - N_\uparrow)$, where $N_\downarrow$ and $N_\uparrow$ are the total number of dipoles that lie parallel and antiparallel to the external field, respectively. The magnetic susceptibility is defined as $\chi(H,T) = \dfrac{\partial \overline{M}}{\partial H}$.

(d) Show that the mean magnetisation and energy are related by $\bar{M} = -N\bar{\epsilon}/H$. **[4]**

(e) Consider the regime where $mH \ll k_B T$ and show that the quantity

$$\chi(0,T) \equiv \lim_{H\to 0} \chi(H,T)$$

satisfies the Curie law $\chi(0,T) \propto 1/(k_B T)$. **[4]**

(f) Explain physically why we expect the susceptibility to diverge in the $T \to 0$ limit. **[3]**

C.2 This question concerns the three-dimensional ideal gas within the semi-classical approximation. The density of states of energy ϵ for a single particle of mass M confined to a volume V is

$$g(\epsilon) = \pi V \left(\frac{M}{\hbar^2 \pi^2}\right)^{\frac{3}{2}} \sqrt{\frac{\epsilon}{2}} \, .$$

The gas is at equilibrium with a reservoir at temperature T.

(a) Explain why, within the semi-classical approximation, the partition function for N weakly-interacting particles is written as

$$Z(N) \approx \frac{[Z(1)]^N}{N!}$$

where $Z(1)$ is the single-particle partition function. **[4]**

(b) By approximating the partition function $Z(1) = \sum_\epsilon \Omega(\epsilon) \mathrm{e}^{-\epsilon/k_B T}$ as an integral involving the density of states, show that

$$Z(1) \approx C \pi V \left(\frac{M k_B T}{\hbar^2 \pi^2}\right)^{\frac{3}{2}}$$

where C is a dimensionless constant *that you do not need to calculate.* **[5]**

(c) Determine the equilibrium Helmholtz free energy of the N-particle gas under the assumption that $N \gg 1$. **[3]**

(d) Given that the mean energy of the ideal gas is $\bar{E} = \frac{3}{2} N k_B T$, show that the entropy is

$$S = Nk \left[\frac{3}{2} \ln \left(\frac{M k_B T}{\hbar^2 \pi^2}\right) - \ln \left(\frac{N}{C \pi V}\right) + \frac{5}{2}\right] \, .$$ **[3]**

(e) Show that the entropy becomes unphysical at a temperature proportional to $\rho^{2/3}$, where $\rho = N/V$. What is the origin of this problem? **[5]**

C.3 The energy levels of a spin-$\frac{1}{2}$ fermion in one dimension are

$$\epsilon_n = \frac{\hbar^2 \pi^2 n^2}{2ML^2}$$

where n is a positive integer, M is the mass of the fermion and L is the length of a line that the fermion is confined to.

(a) Show that the density of states for a spin-$\frac{1}{2}$ fermion in one dimension is

$$g(\epsilon) = \sqrt{\frac{2ML^2}{\hbar^2 \pi^2 \epsilon}} \; .$$ **[5]**

(b) Sketch the $T \to 0$ limiting form of the Fermi-Dirac distribution,

$$f_+(\epsilon) = \frac{1}{\mathrm{e}^{\beta(\epsilon-\mu)} + 1} \; ,$$

noting the position of the Fermi energy ϵ_{f}. **[3]**

(c) By considering the mean number of particles $\bar{N}$ at $T = 0$, show that the Fermi energy for a one-dimensional electron gas is

$$\epsilon_{\mathrm{f}} = \frac{\hbar^2 \pi^2 \bar{N}^2}{8ML^2} \; .$$ **[4]**

(d) Obtain an expression for the mean energy of this gas at zero temperature that contains only ϵ_{f}, $\bar{N}$ and a numerical factor. **[5]**

(e) Explain why the energy of the Fermi gas at low temperatures differs from that of the classical ideal gas in one dimension, which satisfies $\bar{E} = \frac{1}{2}\bar{N}k_B T$. **[3]**