Quantum Mechanics

Assessed problem sheet 2 - to be handed to TO on Fri 26/11 at 12.

Consider a free particle in three dimensions; its dynamics is determined by the Hamiltonian:

$$\hat{H} = \frac{\hat{P}^2}{2\mu}\,, \tag{1}$$

where $\hat{P}^2 = \hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2$, and μ denotes the mass of the particle.

We can define two sets of compatible observables:

$$\mathcal{S}_1 = \left\{\hat{H}, \hat{P}_x, \hat{P}_y, \hat{P}_z, \right\}\,, \tag{2}$$

$$\mathcal{S}_2 = \left\{\hat{H}, \hat{L}^2, \hat{L}_z\right\}\,. \tag{3}$$

1. Check that the observables in $\mathcal{S}_1$ are compatible, and find their simultaneous eigenfunctions. Is the spectrum discrete or continuous? Write the normalization condition for the eigenfunctions.

2. Consider a generic state described by the wave function $\psi(\underline{x})$. Write $\psi(\underline{x})$ as a superposition of the eigenstates found in Q1. Show that the coefficients of such a superposition are given by the Fourier transform $\tilde{\psi}(\underline{p})$ of the wave function. What is the probabilistic interpretation of $\tilde{\psi}(\underline{p})$?

3. Check that the observables in $\mathcal{S}_2$ are compatible. Write the set of equations obeyed by their simultaneous eigenfunctions. In particular find the equation for the radial part of the wave function. Find explicitly the solution for the case $\ell = 0$, specifying carefully the boundary conditions.

4. Consider now the case $\ell \neq 0$. Using the Ansatz:

$$\chi_\ell(r) = r^{\ell+1}\eta_\ell(r)\,, \tag{4}$$

find the differential equation obeyed by $\eta_\ell(0)$.
Check that

$$\eta_\ell = K_\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \frac{\sin(kr)}{kr} \tag{5}$$

where $k = \sqrt{2\mu E/\hbar^2}$, is a solution of the differential equation that you found in the first part of this question.
Taking the proper normalizations into account, the eigenfunctions of the operators in $\mathcal{S}_2$ can be written:

$$\psi_{klm}(r,\theta,\phi) = \sqrt{\frac{2k^2}{\pi}}\, j_\ell(kr) Y_\ell^m(\theta,\phi)\,. \tag{6}$$

The functions j_ℓ are called spherical Bessel functions; their asymptotic behaviour at large distances is

$$j_\ell(x) \sim \frac{\sin(x - \ell\pi/2)}{x}\,. \tag{7}$$

Discuss the physical meaning of this behaviour.

5. A generic state represented by the wave function $\psi(\underline{x})$ can be expanded in the eigenstates with defined angular momentum Eq. (6). Write such an expansion for a plane wave travelling along the z direction with momentum $p = \hbar k$ in terms of unknown coefficients $c(k')_{lm}$. Show that $c(k')_{lm}$ are non-zero only if $k' = k$ and $m = 0$.