Quantum Mechanics

Assessed problem sheet 3 - to be handed to TO by Thurs 8th December 5pm.

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Section A: Answer ALL of the questions in this Section

A.1. The components of the angular momentum operator are given by:

$$\hat{L}_j = \epsilon_{jkl}\hat{x}_k\hat{p}_l\,,$$

where the indices j, k, l range from 1 to 3, and summation over repeated indices is understood. Write the explicit expression for the component $\hat{L}_2$. [1]

Using the canonical commutation relations:

$$\begin{aligned} [\hat{x}_j, \hat{x}_k] &= [\hat{p}_j, \hat{p}_k] = 0, \\ [\hat{x}_j, \hat{p}_k] &= i\hbar\delta_{jk}, \qquad j, k = 1, 2, 3, \end{aligned}$$

prove that $[\hat{L}_2, \hat{L}_3] = i\hbar\hat{L}_1$. [3]

Write the generic expression for the commutator $[\hat{L}_j, \hat{L}_k]$. [1]

A.2. Let $u_n(x)$ be eigenstates of the Hamiltonian $\hat{H}$ with eigenvalues E_n.

Write down the time dependence of a wave function $\Psi(x, t)$ that satisfies the boundary condition $\Psi(x, 0) = u_k(x)$. [2]

If a quantum system is in a state:

$$\psi(x) = \sum_n a_n u_n(x)$$

at time $t = 0$, write down the wave function of the system at time t. [3]

A.3. Consider a three-dimensional harmonic oscillator, whose Hamiltonian is given by:

$$H = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + \frac{1}{2}m\omega^2\left(\hat{x}^2 + \hat{y}^2 + \hat{z}^2\right)\,,$$

what is the degeneracy of the state with energy $E = \frac{5}{2}\hbar\omega$? [5]

A.4. Consider two particles with spin $s_1 = 1/2$ and $s_2 = 1$ respectively. For each particle enumerate the possible values of the spin along the z direction. [2]

Count the number of states in the uncoupled basis $|s_1, s_{1,z}; s_2, s_{2,z}\rangle$. [1]

Is it possible to construct a state with total spin $s = 2$? [2]

Section B: Answer TWO questions from this Section

B.1. Consider a system made of two particles of mass m_1 and m_2 respectively. Let us denote by $\underline{x}_1$ and $\underline{x}_2$ the coordinates of particle 1 and 2 respectively, and let us define:

$$\underline{X} = \frac{m_1\underline{x}_1 + m_2\underline{x}_2}{m_1 + m_2}, \quad \underline{x} = \underline{x}_1 - \underline{x}_2 .$$

(a) Find $\underline{x}_1$ and $\underline{x}_2$ in terms of $\underline{X}$ and $\underline{x}$.

Express the components of the momenta $\hat{P}_k = -i\hbar\frac{\partial}{\partial X_k}$, and $\hat{p}_k = -i\hbar\frac{\partial}{\partial x_k}$ as functions of the components of the momenta of each individual particle $\hat{\underline{p}}^{(1)}$ and $\hat{\underline{p}}^{(2)}$. [3]

(b) Show that

$$\begin{aligned} \hat{\underline{p}}^{(1)} &= \underline{p} + \frac{m_1}{m_1 + m_2}\underline{P}, \\ \hat{\underline{p}}^{(2)} &= -\underline{p} + \frac{m_2}{m_1 + m_2}\underline{P}. \end{aligned}$$

[3]

(c) Show that the total angular momentum of the system can be written as:

$$\hat{\underline{L}} = \hat{\underline{L}}_G + \hat{\underline{L}}_r ,$$

where:

$$\hat{\underline{L}}_r = \hat{\underline{x}} \times \hat{\underline{p}}, \quad \hat{\underline{L}}_G = \hat{\underline{X}} \times \hat{\underline{P}} .$$

Give a physical interpretation of this result. [4]

(d) Consider now two identical particles. How does $\underline{X}$ change when the two particles are exchanged? How does $\underline{x}$ change when the two particles are exchanged? Identify the exchange of the two particles with a parity operation. [5]

(e) Consider now two identical particles of spin $1/2$. What are the possible values of the total spin of the system? Assume that the two identical particles are in a state with $L_r = 1$. Is the spatial wave function of the system symmetric or antisymmetric under the exchange of the two particles? Deduce the value of the total spin in this case. [5]

B.2.

The hydrogen atom is a bound state of an electron and a proton, due to the electromagnetic interaction between the two particles.

(a) Introducing the relative distance between the electron and the proton, write down the Hamiltonian for this system. Discuss the symmetry properties of the Coulomb potential. Explain why we shall look for a solution of the form: $\psi(r,\theta,\phi) = R(r)Y_\ell^m(\theta,\phi)$. [3]

(b) The Laplacian in three dimensions can be written as:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{\hbar^2 r^2}\,.$$

Check the dimensions on both sides of this equation. [1]

Starting from the time-independent Schrödinger equation, and using the fact that the action of $\hat{L}^2$ on the spherical harmonics is given by:

$$\hat{L}^2 Y_\ell^m(\theta,\phi) = \hbar^2\ell(\ell+1)Y_\ell^m(\theta,\phi)\,,$$

write down the one-dimensional differential equation for the radial part of the wave function. [3]

(c) Using a further substitution, show that the equation for the radial part of the wave function can be reduced to a one-dimensional time-independent Schrödinger equation. Interpret the terms that appear in this one-dimensional equation. [3]

(d) The energy of the ground state for the hydrogen atom is given by:

$$E_1 = -\frac{e^2}{2a_0}\,,$$

where $a_0 = 0.53 \times 10^{-10} m$ is the Bohr radius, which describes the typical size of the atom in the ground state.

Write down the excited energy levels, and deduce the typical radius r_k of the excited states for $k > 1$, where k is the principal quantum number. [4]

(e) Consider a sample of hydrogen gas in a lab, at typical values of pressure and temperature, $P = 10^5$ Pa, $T = 300K$. Using the ideal gas law, $PV = nk_BT$, where n is the number of atoms, deduce the volume per atom of the gas. You can use $k_B = 1.38 \times 10^{-23} JK^{-1}$ for Boltzmann's constant. When the typical volume of the hydrogen atom exceeds the available volume per atom, the energy level can no longer be observed in experiments; deduce the maximum principal quantum number that can be observed in the lab conditions above. [6]

B.3. Let us consider an eigenstate $|\psi(t)\rangle$ of an Hamiltonian $\hat{H}$ with eigenvalue E.

(a) Let $\hat{O}$ be a generic operator that does not depend on time. Show that

$$\frac{d}{dt}\langle\psi(t)|\hat{O}|\psi(t)\rangle = 0\,.$$

[5]

(b) Using the fact that $\hat{p} = -i\hbar\,(d/dx)$, show that

$$[\hat{p}, F(\hat{x})] = -i\hbar\frac{d}{dx}F(\hat{x})\,,$$

where $F(x)$ is a differentiable function of x. [5]

(c) Let the Hamiltonian be

$$H = \frac{\hat{p}^2}{2m} + V(\hat{x})\,;$$

compute $\left[\hat{x}\hat{p}, \hat{H}\right]$. [5]

(d) Deduce that

$$\langle\psi(t)|\frac{\hat{p}^2}{2m}|\psi(t)\rangle = 2\langle\psi(t)|V(\hat{x})|\psi(t)\rangle\,,$$

when $V(x) = \lambda x^4$. [5]