Lecture 13

Angular momentum - Reloaded

13.1 Introduction

In previous lectures we have introduced the angular momentum starting from the classical expression:

$$\underline{L} = \underline{r} \times \underline{p}\,, \tag{13.1}$$

and have defined a quantum mechanical operator by replacing $\underline{r}$ and $\underline{p}$ with the corresponding operators: Eq. (13.1) then defines a triplet of differential operators acting on the wave functions. The eigenvalue equations for $\hat{L}^2$ and $\hat{L}_z$ can be written as differential equations, whose solutions yield the eigenvalues and the eigenfunctions of the angular momentum.

In this lecture we are going to follow a different approach, and derive the quantization of angular momentum directly from the commutations relations of the components of $\underline{L}$. This approach is more generic and does not rely on the specific realization of the angular momentum as a differential operator. We will use in this context the symbol $\underline{\hat{J}}$ to denote the angular momentum. Remember that $\underline{\hat{J}}$ is a vector, i.e. it is a triplet of operators. In Cartesian coordinates $\underline{\hat{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$, and the commutation relations are: Recall that the commutation relations are

$$\boxed{[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z \qquad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x \qquad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y}\,. \tag{13.2}$$

The square of the angular momentum is represented by the operator

$$\hat{J}^2 \equiv \hat{J}_x{}^2 + \hat{J}_y{}^2 + \hat{J}_z{}^2, \tag{13.3}$$

with the property that

$$[\hat{J}^2, \hat{J}_x] = [\hat{J}^2, \hat{J}_y] = [\hat{J}^2, \hat{J}_z] = 0. \tag{13.4}$$

The Compatibility Theorem tells us that, for example, the operators $\hat{J}^2$ and $\hat{J}_z$ have simultaneous eigenstates. We denote these common eigenstates by $|\lambda, m\rangle$. Looking back at the results obtained in the previous lectures, these are the kets associated to the sperical harmonics $Y_\ell^m(\theta, \phi)$. We can write:

$$\hat{J}^2\,|\lambda, m\rangle = \lambda\hbar^2\,|\lambda, m\rangle \tag{13.5}$$

$$\hat{J}_z\,|\lambda, m\rangle = m\hbar\,|\lambda, m\rangle \tag{13.6}$$

so that the eigenvalues of $\hat{J}^2$ are denoted by $\lambda\hbar^2$ whilst those of $\hat{J}_z$ are denoted by $m\hbar$.

13.2 The Eigenvalue Spectra of $\hat{J}^2$ and $\hat{J}_z$

We now address the problem of finding the allowed values of λ and m. We introduce *raising and lowering operators*:

$$\boxed{\hat{J}_\pm \equiv \hat{J}_x \pm i\hat{J}_y}\,, \tag{13.7}$$

and calculate the commutators with $\hat{J}^2$ and $\hat{J}_z$. Since $\hat{J}^2$ commutes with both $\hat{J}_x$ and $\hat{J}_y$ we get immediately that

$$[\hat{J}^2, \hat{J}_\pm] = 0\,, \tag{13.8}$$

whilst

$$[\hat{J}_z, \hat{J}_+] = [\hat{J}_z, \hat{J}_x] + i\,[\hat{J}_z, \hat{J}_y] = i\hbar\,\hat{J}_y + i(-i\hbar\,\hat{J}_x) = \hbar\,\hat{J}_+\,, \tag{13.9}$$

and

$$[\hat{J}_z, \hat{J}_-] = [\hat{J}_z, \hat{J}_x] - i\,[\hat{J}_z, \hat{J}_y] = i\hbar\,\hat{J}_y - i(-i\hbar\,\hat{J}_x) = -\hbar\,\hat{J}_-\,. \tag{13.10}$$

The structure of these commutation relations should remind you of the harmonic oscillator problem: we can show that $\hat{J}_+$ and $\hat{J}_-$ act as *raising and lowering operators*, not for the total energy but for the z-component of the angular momentum.

Consider the action of the commutator in Eq. (13.9) on an eigenstate $|\lambda, m\rangle$:

$$[\hat{J}_z, \hat{J}_+]\,|\lambda, m\rangle \equiv \left(\hat{J}_z\hat{J}_+ - \hat{J}_+\hat{J}_z\right)\,|\lambda, m\rangle = \hbar\,\hat{J}_+\,|\lambda, m\rangle\,, \tag{13.11}$$

but we can use the eigenvalue equation $\hat{J}_z\,|\lambda, m\rangle = m\hbar\,|\lambda, m\rangle$ to rewrite this as

$$\boxed{\hat{J}_z\hat{J}_+\,|\lambda, m\rangle = (m+1)\,\hbar\,\hat{J}_+\,|\lambda, m\rangle}\,, \tag{13.12}$$

which says that $\hat{J}_+\,|\lambda, m\rangle$ *is also an eigenstate of* $\hat{J}_z$ *but with eigenvalue* $(m+1)\,\hbar$, *unless* $\hat{J}_+\,|\lambda, m\rangle \equiv 0$. Thus the operator $\hat{J}_+$ acts as a raising operator for the z-component of angular momentum.

Similarly the second commutator can be used to show that

$$\boxed{\hat{J}_z\hat{J}_-\,|\lambda, m\rangle = (m-1)\,\hbar\,\hat{J}_-\,|\lambda, m\rangle}\,, \tag{13.13}$$

which says that $\hat{J}_-\,|\lambda, m\rangle$ *is also an eigenstate of* $\hat{J}_z$ *but with eigenvalue* $(m-1)\,\hbar$, *unless* $\hat{J}_-\,|\lambda, m\rangle \equiv 0$. Thus the operator $\hat{J}_-$ acts as a lowering operator for the z-component of angular momentum.

Notice that since $[\hat{J}^2, \hat{J}_\pm] = 0$ we have

$$\hat{J}^2\left(\hat{J}_\pm\,|\lambda, m\rangle\right) = \hat{J}_\pm\left(\hat{J}^2\,|\lambda, m\rangle\right) = \lambda\hbar^2\left(\hat{J}_\pm\,|\lambda, m\rangle\right)\,, \tag{13.14}$$

so that the states generated by the action of $\hat{J}_\pm$ are still eigenstates of $\hat{J}^2$ *belonging to the same eigenvalue* $\lambda\hbar^2$. Thus we can write

$$\begin{aligned}\hat{J}_+\,|\lambda, m\rangle &= c_+\hbar\,|\lambda, m+1\rangle\\ \hat{J}_-\,|\lambda, m\rangle &= c_-\hbar\,|\lambda, m-1\rangle\end{aligned}$$

where $c_\pm$ are constants of proportionality.

We now observe that, for a given λ, $m^2 \leq \lambda$ so that m must have both a *maximum value*, $m_{\max}$, and a *minimum value*, $m_{\min}$.

Proof:

$$\begin{aligned} \left(\hat{J}^2 - \hat{J}_z{}^2\right) |\lambda, m\rangle &= \left(\hat{J}_x{}^2 + \hat{J}_y{}^2\right) |\lambda, m\rangle \qquad \text{implying that} \\ \left(\lambda - m^2\right) \hbar^2 |\lambda, m\rangle &= \left(\hat{J}_x{}^2 + \hat{J}_y{}^2\right) |\lambda, m\rangle \end{aligned}$$

Taking the scalar product with $\langle\lambda, m|$ yields

$$\left(\lambda - m^2\right) \hbar^2 = \langle \hat{J}_x{}^2 + \hat{J}_y{}^2 \rangle \geq 0\,,$$

so that

$$\lambda - m^2 \geq 0 \qquad \text{or} \qquad -\sqrt{\lambda} \leq m \leq \sqrt{\lambda}\,. \tag{13.15}$$

Hence the spectrum of $\hat{J}_z$ is bounded above and below, for a given λ. We can deduce that

$$\begin{aligned} \hat{J}_+ |\lambda, m_{\max}\rangle &= 0\,, \qquad \text{and} \\ \hat{J}_- |\lambda, m_{\min}\rangle &= 0 \end{aligned}$$

To proceed further, we need a couple of identities for $\hat{J}^2$ which follow from the definitions of $\hat{J}_\pm$. Noting that

$$\begin{aligned} \hat{J}_+\hat{J}_- &= \hat{J}_x{}^2 + \hat{J}_y{}^2 + i\hat{J}_y\hat{J}_x - i\hat{J}_x\hat{J}_y = \hat{J}_x{}^2 + \hat{J}_y{}^2 + \hbar\hat{J}_z\,, \\ \hat{J}_-\hat{J}_+ &= \hat{J}_x{}^2 + \hat{J}_y{}^2 - i\hat{J}_y\hat{J}_x + i\hat{J}_x\hat{J}_y = \hat{J}_x{}^2 + \hat{J}_y{}^2 - \hbar\hat{J}_z\,, \end{aligned}$$

we can write

$$\boxed{\begin{aligned} \hat{J}^2 &\equiv \hat{J}_+\hat{J}_- - \hbar\hat{J}_z + \hat{J}_z{}^2 \qquad \text{or, equally,} \\ \hat{J}^2 &\equiv \hat{J}_-\hat{J}_+ + \hbar\hat{J}_z + \hat{J}_z{}^2 \end{aligned}}\,. \tag{13.16}$$

Applying the first of these to the state of minimum m, we find

$$\begin{aligned} \hat{J}^2 |\lambda, m_{\min}\rangle &= \left(\hat{J}_+\hat{J}_- - \hbar\hat{J}_z + \hat{J}_z{}^2\right) |\lambda, m_{\min}\rangle \\ &= \left(-m_{\min}\hbar^2 + m_{\min}^2\hbar^2\right) |\lambda, m_{\min}\rangle\,, \qquad \text{since } \hat{J}_- |\lambda, m_{\min}\rangle = 0 \\ &= m_{\min}(m_{\min} - 1)\hbar^2 |\lambda, m_{\min}\rangle \\ &\equiv \lambda\hbar^2 |\lambda, m_{\min}\rangle\,. \end{aligned}$$

Thus we deduce that

$$\lambda = m_{\min}(m_{\min} - 1)\,.$$

Similarly, using the second of the two identities for $\hat{J}^2$

$$\begin{aligned}
\hat{J}^2 |\lambda, m_{\max}\rangle &= \left(\hat{J}_-\hat{J}_+ + \hbar\hat{J}_z + \hat{J}_z{}^2\right) |\lambda, m_{\max}\rangle \\
&= \left(m_{\max}\hbar^2 + m_{\max}^2\hbar^2\right) |\lambda, m_{\max}\rangle\,, \qquad \text{since } \hat{J}_+ |\lambda, m_{\max}\rangle = 0 \\
&= m_{\max}(m_{\max}+1)\hbar^2 |\lambda, m_{\max}\rangle \\
&\equiv \lambda\hbar^2 |\lambda, m_{\max}\rangle
\end{aligned}$$

and hence we obtain a second expression for λ:

$$\lambda = m_{\max}(m_{\max}+1)\,.$$

Usually, $m_{\max}$ is denoted by j and so we write this as

$$\lambda = j(j+1) = m_{\min}(m_{\min}-1)\,.$$

This is a quadratic equation for $m_{\min}$:

$$m_{\min}^2 - m_{\min} - j^2 - j = 0\,,$$

which can be factorised:

$$(m_{\min}+j)(m_{\min}-j-1) = 0\,,$$

and we see that, since $m_{\min} \leq j$ by definition, the only acceptable root is

$$m_{\min} = -j\,.$$

Now since $m_{\max}$ and $m_{\min}$ differ by some integer, k, say, we can write

$$m_{\max} - m_{\min} = k, \qquad k = 0,\, 1,\, 2,\, 3, \ldots\,,$$

or

$$j - (-j) \equiv 2j = k, \qquad k = 0,\, 1,\, 2,\, 3, \ldots\,,$$

so that the allowed values of j are

$$\boxed{j = 0,\, \tfrac{1}{2},\, 1,\, \tfrac{3}{2},\, 2, \ldots}\,. \tag{13.17}$$

For a given value of j, we see that m ranges over the values

$$\boxed{j,\, j-1,\, \ldots,\, -j+1,\, -j} \tag{13.18}$$

a total of $(2j+1)$ values.

13.3 Nomenclature

From the results presented above we can draw the following conclusions.

- The eigenvalues of $\hat{J}^2$ are $j(j+1)\hbar^2$, where j is one of the allowed values

$$\boxed{j = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ 2, \ldots}\,.$$

- Since $\lambda = j(j+1)$, we can equally well label the simultaneous eigenstates of $\hat{J}^2$ and $\hat{J}_z$ by j rather than by λ so that

$$\boxed{\begin{array}{rcll} \hat{J}^2\,|j,m\rangle &=& j(j+1)\hbar^2\,|j,m\rangle & \text{and} \\ \hat{J}_z\,|j,m\rangle &=& m\hbar\,|j,m\rangle & \end{array}}\,.$$

- For a given value of j, there are $(2j+1)$ possible eigenvalues of $\hat{J}_z$, denoted by $m\hbar$, where m runs from j to $-j$ in integer steps.

- The set of $(2j+1)$ states $\{|j,m\rangle\}$ is called a *multiplet*.

13.4 Normalization

We need to determine the constants of proportionality $c_\pm$ that appear in the equations

$$\boxed{\begin{array}{rcl} \hat{J}_+\,|j,m\rangle &=& c_+\hbar\,|j,m+1\rangle \\ \hat{J}_-\,|j,m\rangle &=& c_-\hbar\,|j,m-1\rangle \end{array}}\,.$$

Let us consider

$$\begin{aligned} \langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle &= c_+\hbar\,\langle j,m|\hat{J}_-|j,m+1\rangle \\ &= c_+\hbar\,\langle j,m+1|(\hat{J}_-)^\dagger|j,m\rangle^*\,, \quad \text{from the definition of } \dagger \\ &= c_+\hbar\,\langle j,m+1|\hat{J}_+|j,m\rangle \qquad \text{since } (\hat{J}_-)^\dagger = \hat{J}_+ \\ &= c_+c_+^*\hbar^2\,\langle j,m+1|j,m+1\rangle \\ &= |c_+|^2\hbar^2\,, \qquad \text{from orthonormality}\,. \end{aligned}$$

But we can evaluate the left hand side by making use of the identity Eq. (13.16):

$$\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^{\ 2} - \hbar\hat{J}_z\,,$$

yielding

$$\begin{aligned}
\langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle &= \langle j,m|\hat{J}^2 - \hat{J}_z{}^2 - \hbar\hat{J}_z|j,m\rangle \\
&= \langle j,m|j(j+1)\hbar^2 - m^2\hbar^2 - m\hbar^2|j,m\rangle \\
&= [j(j+1) - m(m+1)]\hbar^2\,, \qquad \text{from orthonormality}\,.
\end{aligned}$$

Thus we obtain

$$\boxed{|c_+|^2 = j(j+1) - m(m+1)}\,.$$

In similar fashion,

$$\begin{aligned}
\langle j,m|\hat{J}_+\hat{J}_-|j,m\rangle &= c_-\hbar\,\langle j,m|\hat{J}_+|j,m-1\rangle \\
&= c_-\hbar\,\langle j,m-1|(\hat{J}_+)^\dagger|j,m\rangle^*\,, \qquad \text{from the definition of } \dagger \\
&= c_-\hbar\,\langle j,m-1|\hat{J}_-|j,m\rangle\,, \qquad \text{since } (\hat{J}_+)^\dagger = \hat{J}_- \\
&= c_-c_-^*\hbar^2\,\langle j,m-1|j,m-1\rangle \\
&= |c_-|^2\hbar^2\,, \qquad \text{from orthonormality}\,.
\end{aligned}$$

together with the other identity

$$\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z{}^2 + \hbar\hat{J}_z\,,$$

yields

$$\boxed{|c_-|^2 = j(j+1) - m(m-1)}\,.$$

Condon-Shortley Phase Convention

Taking $c_\pm$ to be real and positive gives

$$\boxed{c_\pm = \sqrt{j(j+1) - m(m\pm1)}}\,. \tag{13.19}$$

13.5 Matrix Representations

For a given j, the quantities $\langle j,m'|\hat{J}_z|j,m\rangle$ are known as the *matrix elements* of $\hat{J}_z$. We can calculate what they are as follows:

$$\boxed{\langle j,m'|\hat{J}_z|j,m\rangle = m\hbar\,\langle j,m'|j,m\rangle = m\hbar\,\delta_{m',m}}\,, \tag{13.20}$$

where we have used the orthonormality properties of the basis. Why matrix elements? As we have already seen, we can regard the labels m' and m as *labelling the rows and columns,* *respectively, of a matrix*. Given that m' and m can each take $(2j+1)$ values, the matrix in question is $(2j+1) \times (2j+1)$.

Similarly, the matrix elements of the raising and lowering operators are given by

$$\boxed{\langle j, m' | \hat{J}_{\pm} | j, m \rangle = c_{\pm} \hbar \, \langle j, m' | j, m \pm 1 \rangle = \sqrt{j(j+1) - m(m \pm 1)} \, \hbar \, \delta_{m', m \pm 1}}$$

Check that for $j = 1$ you recover the results obtained in Q4 of Problem Sheet 5.

13.6 Summary

As usual, we summarize the main concepts introduced in this lecture.

- Properties of the angular momentum from commutation relations.
- Derivation of the eigenvalue spectrum.
- Eigenstates and normalization.
- Matrix representations.