Lecture 9

Angular momentum - 2

9.1 Spectroscopic Notation

We saw in the last section that the only allowed values of the angular momentum quantum number, ℓ, are

$$\ell = 0,\, 1,\, 2,\, 3, \dots$$

For historical reasons, each possible ℓ value is denoted by a letter of the alphabet according to the following sequence:

$$s,\ p,\ d,\ f,\ g,\ h,\ \dots \text{ and so on alphabetically}$$

Thus, for example, an $\ell = 0$ state is referred to as an s-wave state, and so on. The reason for the non-alphabetic order of the first four letters is that they are taken from the names of four series of spectral lines, called s*harp,* p*rincipal,* d*iffuse and* f*ine*, observed in the spectra of alkali atoms such as Sodium.

ℓ	0	1	2	3	4	5
notation	s	p	d	f	g	h

Table 9.1: Spectroscopic notation for the first few ℓ values.

9.2 Dirac's notation

We can introduce a Dirac notation for the spherical harmonics. To each function $Y_\ell^m(\theta, \phi)$ we associate a ket $|\ell, m\rangle$, which represents a vector in an infinite-dimensional space.

The scalar product of two vectors can be written using the bra-ket notation:

$$\langle \ell', m' | \ell, m \rangle = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, Y_{\ell'}^{m'}(\theta, \phi)^*\, Y_\ell^m(\theta, \phi)\,. \tag{9.1}$$

The eigenvalue equations for $\hat{L}^2$ and $\hat{L}_z$ can be written using Dirac's notation:

$$\hat{L}_z |\ell, m\rangle = \hbar m |\ell, m\rangle\,, \tag{9.2}$$

$$\hat{L}^2 |\ell, m\rangle = \hbar^2 \ell(\ell+1) |\ell, m\rangle\,. \tag{9.3}$$

The orthonormality of the spherical harmonics is concisely written as:

$$\langle \ell', m' | \ell, m \rangle = \delta_{\ell', \ell} \delta_{m', m}\,. \tag{9.4}$$

The completeness of the spherical harmonics amounts to the fact that *any* function $f(\theta, \phi)$ has a unique expansion:

$$f(\theta, \phi) = \sum_{\ell, m} c_{\ell, m} Y_\ell^m(\theta, \phi)\,, \tag{9.5}$$

where:

$$c_{\ell,m} = \langle \ell, m | f \rangle \,. \tag{9.6}$$

9.3 Experimental Evidence for Quantisation

Since the results that we have just obtained seem so bizarre, let us consider the experimental evidence supporting the contention that the component of angular momentum in a particular direction, say the z-direction, cannot take any old value but is restricted to be one of $(2\ell+1)$ possible values.

We will consider in barest outline an experiment known as the Stern-Gerlach experiment, in which a beam of neutral paramagnetic particles (atoms) is deflected by a inhomogeneous magnetic field. The apparatus is schematically represented in Fig. 9.1. The inhomogeneous magnetic field between the poles sketched in the figure causes each particle in the beam to be deflected by an amount proportional to its magnetic moment. The original experiment was first performed by Stern and Gerlach in 1922 in Frankfurt. Note that Stern at that time was an assistant to Max Born, before the latter came to Edinburgh.

Let the vertical direction be the z-direction, then the deflection at the screen is proportional to the z-component of the magnetic moment, which in turn is proportional to the z-component of the angular momentum of each particle. Thus the deflection is a measure of the observable L_z.

In classical mechanics we would expect a continous range of deflections, corresponding to the random orientations of the magnetic moments of the particles emerging from the oven. Instead the beam appears to be split into a number of discrete components. This phenomenon is sometimes referred to as *spatial quantisation*. The number of beams is $(2\ell + 1)$, so that if the theory discussed so far is correct there should always be an odd number of beams.

Actually, the original experiment of Stern and Gerlach observed just two beams, which would correspond to $\ell = \frac{1}{2}$, not one of the allowed values! We shall return to discuss the significance of this observation later. In the meantime we simply note that the experiment supports the idea that the component of angular momentum in a given direction can only take certain discrete values.

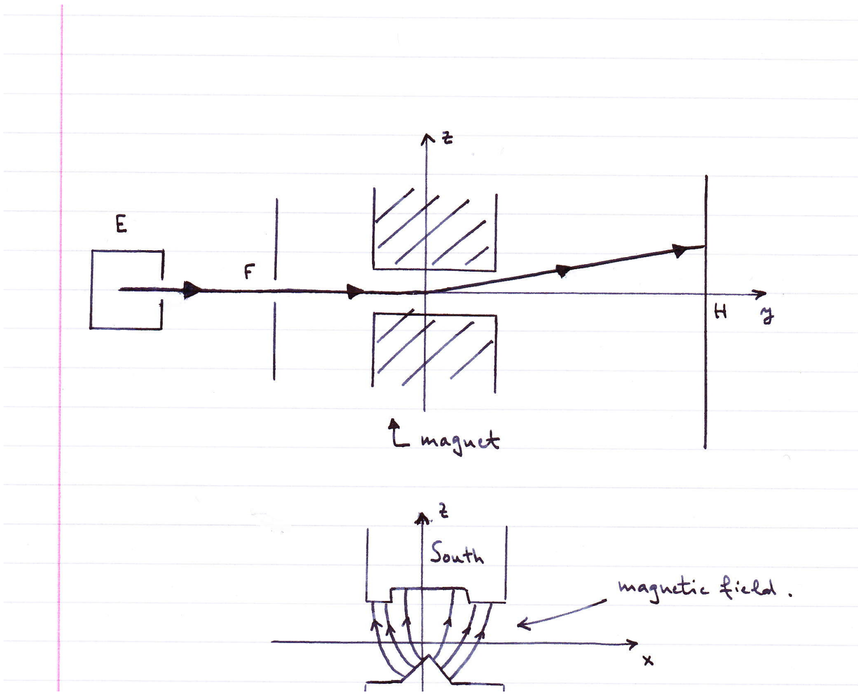

Figure 9.1: A beam of atoms leaves the oven E, is collimated by the slit F and is split by the inhomogeneous magnetic field between the pole pieces, before being detected at the screen. The deflection of the beam depends on the angular momentum of the particles.

9.4 Summary

As usual, we summarize the main concepts introduced in the last two lectures.

- Definition of the angular momentum. Angular momentum as a differential operator.
- Commutation relations between the components of the angular momentum.
- Square of the angular momentum, commutation relations.
- Eigenvalue problem for $\hat{L}^2$ and $\hat{L}_z$.
- Spectroscopic notation.
- Experimental evidence.