Quantum Mechanics

Problem Sheet 3

Basics

1. In this problem you derive an uncertainty principle that is valid for any two non-compatible observables. Check that you get the Heisenberg uncertainty principle if $\hat{A} = \hat{X}$ and $\hat{B} = \hat{P}$.

2. Study of the symmetry properties of the solutions of the TISE.

3. More on the time evolution of the eigenstates of the Hamiltonian. Relation between wave functions at different times.

4. Transmission amplitude for the tunneling particle.

Further problems

1. More on one-dimensional potential wells.

2. Delta function potential.

Basics

1. *Generalized uncertainty principle*
   If ΔA and ΔB denote the uncertainties in the observables $\mathcal{A}$ and $\mathcal{B}$ respectively in the state $\Psi(x,t)$ then the generalised uncertainty relation states that
   $$\Delta A\, \Delta B \geq \tfrac{1}{2} \,||\langle[\hat{A},\hat{B}]\rangle||\,.$$
   In order to prove this relation consider the operators:
   $$\hat{X} = \hat{A} - \langle\hat{A}\rangle$$
   $$\hat{Y} = \hat{B} - \langle\hat{B}\rangle\ .$$
   The uncertainties in the observables are given by:
   $$(\Delta A)^2 = \langle\hat{X}^2\rangle\,,$$
   $$(\Delta B)^2 = \langle\hat{Y}^2\rangle\,.$$
   For any real number λ, we can construct the state:
   $$\Phi(x,t) = \hat{X}\Psi(x,t) + i\lambda\hat{Y}\Psi(x,t)\,.$$
   The norm of the state Φ is positive by definition:
   $$||\Phi||^2 = \int dx\, \Phi(x,t)^*\Phi(x,t)\,.$$
   Use this fact to prove the generalized uncertainty relation.

2. Show that for symmetric potentials, for which $V(-x) = V(x)$, the energy eigenfunctions, which satisfy the time-independent Schrödinger equation, must have definite parity i.e. be either even or odd functions of x. [Hint: make the substitution $x \to -x$ in the Schrödinger equation].

3. Let us consider a complete set of eigenstates of the energy for the case of a discrete spectrum:
   $$\hat{H}u_n(x) = E_n u_n(x), \qquad n \text{ integer}\,.$$
   Write down the expansion of a generic state at time $t = 0$, $\Psi(x,0)$ in the basis of the energy eigenstates. Write down an explicit expression for the coefficients of the expansion.
   Deduce an expression for the wave function $\Psi(x,t)$ at time t.
   Show that:
   $$\Psi(x,t) = \int dx' \left[\sum_n e^{-iE_n t/\hbar} u_n(x) u_n(x')^*\right] \Psi(x',0)\,.$$
   Verify that:
   $$\Psi(x,t) = \int dx'\, K(xt, x't')\Psi(x',t')\,,$$
   where
   $$K(xt,x't') = \sum_n e^{-iE_n(t-t')/\hbar} u_n(x) u_n(x')^*\,.$$

4. Using the continuity equations for the wave function and its derivative at $x = 0$ and $x = a$, compute the transmission amplitude $S(E)$ for the tunneling particle discussed in the lecture notes.

Further problems

1. The continuity equations for the finite well yield:

$$p \tan\left(\frac{pa}{2\hbar}\right) = \bar{p}\,,$$
$$p \cot\left(\frac{pa}{2\hbar}\right) = -\bar{p}\,,$$

respectively in the even and odd parity sectors. Let us define new variables :

$$\xi = pa/(2\hbar)\,,$$
$$\eta = \bar{p}a/(2\hbar)\,.$$

Check that the new variables are dimensionless.
Check that the continuity equations become:

$$\eta = \xi \tan(\xi)\ ,$$
$$\eta = -\xi \cot(\xi)\ ,$$

and that:

$$\xi^2 + \eta^2 = \frac{ma^2 V_0}{2\hbar^2} \equiv R^2\,.$$

Find a graphical solution to this set of equations in the (ξ, η) plane. Discuss the number of solutions as R is varied. Discuss the limit when $a \to 0$, $V_0 \to \infty$, and $aV_0 = g$ is kept fixed.

2. Let us consider now the following potential:

$$V(x) = -g\delta(x)\,, g > 0\,.$$

This is an attractive potential, and we are going to look for bound states, i.e. states with $E < 0$.

The continuity condition for the wave function reads

$$\lim_{x\to 0^-} \psi(x) = \lim_{x\to 0^+} \psi(x)\,.$$

Write down the Schrödinger equation, and integrate it over dx in the interval $[-\varepsilon, \varepsilon]$. Taking the limit $\varepsilon \to 0$, show that:

$$\lim_{x\to 0^+} \psi'(x) - \lim_{x\to 0^-} \psi'(x) = -\frac{2mg}{\hbar^2}\psi(0)\,.$$

Normalizable solutions have the form

$$\psi(x) = \begin{cases} e^{-\kappa x}, & \text{for } x > 0\,, \\ e^{\kappa x}, & \text{for } x < 0\,, \end{cases}$$

with $\kappa = \sqrt{-2mE}/\hbar$. Show that the continuity equation for the derivative yields:

$$E = -\frac{mg^2}{2\hbar^2}\,.$$