Quantum Mechanics

Problem Sheet 4

Basics

1. Separation of variables using Cartesian coordinates in two dimensions.
2. Commutation relations involving the angular momentum. Use the canonical commutation relations.
3. Tedious, but useful.
4. This is something that you need to do at least once... Make sure you are familiar with the manipulations involving spherical coordinates. You should be able to visualize the geometrical origin of the equations that appear in this exercise.
5. Use the expression for the angular momentum in spherical coordinates.
6. Similar to the previous one. Useful practice.

All problems this week are "basics". Make sure you are able to solve most of them. You should definitely be confortable with the first five problems.

Basics

1. Find the energy eigenstates for a particle in a two-dimensional square infinite potential well of size $a \times a$.
   Discuss the degeneracy of the first two energy levels.

2. Compute the commutation relations of the momentum operator $\underline{\hat{P}}$ and the angular momentum $\underline{\hat{L}}$.

3. Using the canonical commutation relations compute

$$\left[\hat{L}_i, \hat{L}_j\right] . \tag{1}$$

4. The gradient operator in spherical polar coordinates is $\vec{\nabla} = \hat{r}\partial/\partial r + \hat{\theta}\partial/r\partial\theta + \hat{\phi}\partial/r\sin\theta\partial\phi$.

   (a) Derive the following entries in the table (e.g. $\partial\hat{r}/\partial\theta = \hat{\theta}$):

	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$
$\partial/\partial r$	0	0	0
$\partial/\partial\theta$	$\hat{\theta}$	$-\hat{r}$	0
$\partial/\partial\phi$	$\hat{\phi}\sin\theta$	$\hat{\phi}\cos\theta$	$-(\hat{r}\sin\theta + \hat{\theta}\cos\theta)$

   (b) Calculate ∇^2 in spherical polar coordinates.

   (c) Calculate L_x, L_y and L_z in spherical polar coordinates.

   (d) Calculate L^2 in spherical polar coordinates.

5. A particle has a wavefunction $u(x, y, z) = A\, z \exp[-b(x^2 + y^2 + z^2)]$, where b is a constant.

   (a) Show that this wavefunction is an eigenfunction of $\hat{L}^2$ and of $\hat{L}_z$ and find the corresponding eigenvalues.
   Hint: use the spherical polar expressions for $\hat{L}^2$ and $\hat{L}_z$, and write the wavefunction in spherical polars.

   (b) Can you identify the physical system for which this is an energy eigenstate ?

6. The wavefunction of a particle is known to have the form

$$u(r, \theta, \phi) = A\, R(r)\, f(\theta)\, \cos 2\phi$$

   where f is an unknown function of θ. What can be predicted about the results of measuring

   (a) the z-component of angular momentum;

   (b) the square of the angular momentum ?

   Hint: Note that $\cos 2\phi = \{\exp(2i\phi) + \exp(-2i\phi)\}/2$.

   Answer the same questions for the wavefunction

$$u(r, \theta, \phi) = A\, R(r)\, f(\theta)\, \cos^2\phi$$