Quantum Mechanics

Problem Sheet 6

Basics

1. polar coordinates in two dimensions.
2. more on the harmonic oscillator.
3. a few details on the Stern-Gerlach experiment.
4. more on S-G.
5. useful problem to get some practice with the eigenstates of the H atom.
6. more practice with the H atom.

Further problems

1. H atom as a harmonic oscillator.
2. Lenz vector - you already saw this in dynamics! Compare with the classical case.

Basics

1. Consider a two-dimensional system. The position of the particle can be described equivalently by the Cartesian coordinates (x, y), or the polar ones (r, θ).
   Sketch a two-dimensional plane, and identify the Cartesian and polar coordinates of a point P.
   Find the relation between the Cartesian and the polar coordinates.
   The gradient operator in two-dimensional polar coordinates is:
   $$\underline{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\,, \tag{1}$$
   use this expression to compute the Laplacian in polar coordinates.
   In two dimensions we can define the angular momentum $\hat{L} = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x$. Note that the angular momentum has only one component in this case!! Compute $\hat{L}$ in polar coordinates.
   Show that:
   $$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\hat{L}^2}{\hbar^2 r^2}\,. \tag{2}$$

2. Write down the time-independent Schrödinger equation for the two-dimensional harmonic oscillator of frequency ω. Find the eigenvalues, eigenfunctions, and their degeneracy by separating the Cartesian coordinates x, y.
   Use the result in **B**1 to write the equation for the radial wave function of a stationary state of the two-dimensional harmonic oscillator.
   (*Hint:* there are several steps to be performed to get to the radial equation. Don't give up!)

3. In an experiment of the Stern-Gerlach type, a beam of atoms, each with $\ell = 1$ and kinetic energy 3×10^{-20} J, travels for 0.03 m through a magnetic field whose gradient $\partial B/\partial z = 2.3 \times 10^3$ Tm^{-1}. Given that the magnitude of the deflecting force on each atom is
   $$F_z = \mu_z \frac{\partial B}{\partial z} \quad \text{and that} \quad \mu_z = \mu_B\, L_z/\hbar$$
   where μ_B, the so-called Bohr magneton, is 9.27×10^{-24} JT^{-1}, calculate the transverse deflections produced at a screen 0.25 m from the exit from the magnet.

4. Can the Stern-Gerlach experiment be done with free electrons? What effect does the proton have in a hydrogen Stern-Gerlach experiment? Give quantitative answers.

5. Consider a hydrogen atom whose wavefunction at time $t = 0$ is the following superposition of energy eigenfunctions;
   $$\Psi(\underline{r}, t) \equiv \psi(\underline{r}) = \frac{1}{\sqrt{78}}\left[2u_{100}(\underline{r}) - 7u_{200}(\underline{r}) + 5u_{322}(\underline{r})\right]$$
   - Is this wavefunction an eigenfunction of the parity operator?
   - What is the probability of obtaining the result (i) E_1 (ii) E_2 (iii) E_3, on measuring the total energy?
   - What are the expectation values of the energy, of the square of the angular momentum and of the z component of the angular momentum ?

6. The normalised energy eigenfunction of the ground state of the hydrogen atom ($Z = 1$) is

$$u_{100}(\underline{r}) = C \exp(-r/a_0)$$

where a_0 is the Bohr radius and C is a normalisation constant. For this state

- calculate the normalisation constant, C; you may wish to note the useful integral

$$\int_0^\infty \exp(-br)\, r^n \,\mathrm{d}r = n!/b^{n+1}, \qquad n > -1$$

- determine the radial distribution function, $D_{10}(r)$, and sketch its behaviour; hence determine the most probable value of the radial coordinate, r, and the probability that the electron is within a sphere of radius a_0; recall that $Y_0^0(\theta, \phi) = 1/\sqrt{4\pi}$;
- calculate the expectation value of r;
- calculate the expectation value of the potential energy, $V(r)$;
- calculate the uncertainty, Δr, in r.

Further problems

1. In the TISE for the H atom, set:

$$r = \lambda z^2/2\,, \tag{3}$$

$$\frac{\chi(r)}{r} = \frac{F(z)}{z}\,; \tag{4}$$

Show that $F(z)$ obeys the radial equation of the two-dimensional harmonic oscillator discussed in **B**2.
Deduce the wave function for the ground state of the H atom.
(*This solution of the H atom is due to Schwinger*).

2. In three dimensions, consider the *Lenz* vector:

$$\underline{\hat{Q}} = \frac{1}{2m}\left(\underline{\hat{P}} \times \underline{\hat{L}} - \underline{\hat{L}} \times \underline{\hat{P}}\right) - \frac{e^2}{r}\underline{\hat{X}}\,, \tag{5}$$

where X and P are the three-dimensional position and momentum, L is the angular momentum operator, and m the mass of a particle. Show that Q commutes with the Hamiltonian for the H atom.
Show that

$$\hat{Q}^2 = e^4 + \frac{2\hat{H}(\hat{L}^2 + \hbar^2)}{m}\,. \tag{6}$$