## Quantum Mechanics

## Problem Sheet 7

1. Write the position operator $\hat{X}$ as a function of the raising and lowering operators $\hat{a}$ and $\hat{a}^{\dagger}$.
Deduce the matrix element of $\hat{X}$ between two eigenstates of the Hamiltonian: $\langle n| \hat{X}|m\rangle$. Repeat the same computations for the momentum operator $\hat{P}$.
2. Consider a one-dimensional harmonic oscillator, consisting of a particle of mass $m$ in a quadratic potential characterized by the frequency $\omega$. Assume that the particle has a charge $q$ and is placed in a uniform electric field $\mathcal{E}$ in the $x$ direction. The Hamiltonian can be written as:

$$
\hat{H}(\mathcal{E})=\frac{\hat{P}^{2}}{2 m}+\frac{1}{2} m \omega^{2} \hat{X}^{2}-q \mathcal{E} \hat{X}
$$

Write the eigenvalue equation for $\hat{H}(\mathcal{E})$.
Replacing the variable $x$ by a new variable $t$ :

$$
t=x-\frac{q \mathcal{E}}{m \omega^{2}}
$$

find the eigenvalues $E_{n}^{\prime}$ and the eigenfunctions $u_{n}^{\prime}(x)$ that solve the equation above.
3. The contribution of the electron to the electric dipole moment of an atom is given by the operator:

$$
\hat{D}=q \hat{X}
$$

In the absence of an electric field, use the answer in Q1 to deduce:

$$
\langle\hat{D}\rangle=q\langle n| \hat{X}|n\rangle=0 .
$$

Let us assume that the electric field is turned on. Compute the mean dipole moment:

$$
\langle\hat{D}\rangle^{\prime}=q\left\langle n^{\prime}\right| \hat{X}\left|n^{\prime}\right\rangle,
$$

where $\left|n^{\prime}\right\rangle$ are the kets associated to $u_{n}^{\prime}(x)$, the wave functions found in Q 2 . Discuss the physical interpretation of this result.
4. Use raising and lowering operators to solve the two-dimensional harmonic oscillator. Discuss the energy levels and their degeneracy.
5. Muonic atom. The muon is a particle with the same properties as the electron except that its mass is 207 times heavier: $m_{\mu} / m_{e}=207$. Its interactions with an atomic nucleus are essentially electromagnetic. A muon can be attracted by the Coulomb field of a nucleus and form a bound states, called a "muonic atom".
Describe the energy of the bound states of a muon in the Coulomb field of a heavy atom such as lead, with charge $Z=82$, in the approximation where the nucleus is considered to be infinitely heavy.
Determine the Bohr radius for this system, and discuss the validity of the computation above, knowing that the radius of the lead nucleus is $\rho_{0} \approx 8.5 \times 10^{-13} \mathrm{~cm}$.

Assuming that the charge is distributed uniformly inside a sphere of radius $\rho_{0}$, the potential inside the atom is:

$$
V(r)=\frac{Z e^{2}}{2 \rho_{0}^{3}} r^{2}+C .
$$

Find the constant $C$ such that the potential is continuous at the boundary of the nucleus. Neglecting the Coulomb field at large $r$, find the energy of the ground state for this model.
6. Let us consider a one-dimensional harmonic oscillator. By recalling the action of the lowering operator $a$ on the ground state, find the first-order differential equation that is satisfied by the wave function $u_{0}(x)$.
Using the creation operator build the wave function for the first two excited states.
7. By considering the action of $\hat{J}_{+}$on the state $|j, j\rangle$, find the first-orer differential equation satisfied by the spherical harmonics $Y_{\ell}^{\ell}(\theta, \phi)$. Deduce an expression for the associated Legendre polynomials.
8. Let us consider the one-dimensional harmonic oscillator. Express $\hat{X}^{2}$ and $\hat{P}^{2}$ in terms of $a$ and $a^{\dagger}$, and show that for an energy eigenstate:

$$
\langle n| \frac{\hat{P}^{2}}{2 m}|n\rangle=\langle n| \frac{1}{2} m \omega^{2} \hat{X}^{2}|n\rangle=\frac{1}{2}\langle n| \hat{H}|n\rangle .
$$

Show that this result remains true when the oscillator is not in an eigenstate of the Hamiltonian, provided a time average over one classical oscillator period is taken. The time average is defined as:

$$
\bar{f}=\frac{1}{T} \int_{0}^{T} d t f(t)
$$

