Quantum Mechanics

Problem Sheet 8

1. The Hamiltonian that describes the interaction of a static spin-$\frac{1}{2}$ particle with an external magnetic field, $\underline{B}$, is
$$\hat{H} = -\hat{\underline{\mu}}.\underline{B}$$
where the magnetic moment operator, $\hat{\underline{\mu}}$, is related to the spin operator, $\hat{\underline{s}}$, by $\hat{\underline{\mu}} = \gamma\, \hat{\underline{s}}$, with γ the gyromagnetic ratio. Use the Pauli representation of the spin vector $\hat{\underline{s}}$ to find the energy eigenvalues in a static uniform magnetic field in the z-direction, $\underline{B}_0 = (0, 0, B_0)$.

2. Construct the matrix $\underline{\sigma}\cdot\underline{e}$, where $\underline{e}$ is a unit vector with Cartesian components e_x, e_y, e_z. $\underline{\sigma}$ has Cartesian components which are the Pauli matrices, σ_x, σ_y, σ_z, so that
$$\underline{\sigma}\cdot\underline{e} = e_x\sigma_x + e_y\sigma_y + e_z\sigma_z.$$
Show that the eigenvalues of $\underline{\sigma}\cdot\underline{e}$ are ± 1 and hence deduce that a measurement of the component of spin along the direction of $\underline{e}$, of a spin-$\frac{1}{2}$ particle can only yield the result $\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$. Obtain the *normalized* eigenvectors of $\underline{\sigma}.\underline{e}$ corresponding to each of the eigenvalues and verify the your answers by considering the special case $e_x = 1$, $e_y = e_z = 0$ considered in lectures. Remember that the normalisation condition for a 2–component column matrix with entries ψ_1 and ψ_2 is
$$|\psi_1|^2 + |\psi_2|^2 = 1.$$

3. A beam of spin-$\frac{1}{2}$ particles is sent through a Stern-Gerlach apparatus which divides the incident beam into two spatially-separated beams having $m = \pm\frac{1}{2}$ respectively. The beam with $m = -\frac{1}{2}$ is removed, whilst the beam with $m = \frac{1}{2}$ is allowed to impinge on a second Stern-Gerlach apparatus whose mean field is also perpendicular to the beam direction, but inclined at an angle θ with respect to that of the first apparatus. Calculate the relative intensities of the two emergent beams.

4. Show that the matrices
$$M_1 = \hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad M_2 = \hbar\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad M_3 = \hbar\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
satisfy the angular momentum algebra. Find the eigenvalues of M_3 and of $M^2 \equiv M_1^2 + M_2^2 + M_3^2$ and hence show that the matrices may be used to represent the components of angular momentum for a system with $j = 1$.

5. Two indistinguishable, non-interacting, spin-$\frac{1}{2}$ particles move in the potential
$$V(x) = \infty, \quad |x| > L/2, \quad V(x) = 0 \quad |x| \leq L/2$$
Obtain expressions for the energy eigenvalues and eigenfunctions of the ground state and first excited state of the two-particle system. What is the degeneracy of the first excited state?

   How would your answers change if the particles in question were spinless?

6. What are the allowed values of the total angular momentum quantum number, j, for a particle with spin $s = \frac{1}{2}$ and orbital angular momentum $\ell = 2$? If two spinless particles each have orbital angular momentum $\ell = 1$, what are the allowed values of the total orbital angular momentum ?

7. Given a system of two non-interacting particles with orbital angular momentum $\ell = 1$, $|m_1, m_2\rangle$ denotes a state where particles 1 and 2 have L_z components $m_1\hbar$ and $m_2\hbar$ respectively.

   Construct the operators $\hat{L}^2$ and $\hat{L}_z$ for the system in terms of the operators $\hat{L}_{z1,2}$ and $\hat{L}_{\pm 1,2}$.

   Normalize the following wave functions. Are these states eigenstates of $\hat{L}^2$ and $\hat{L}_z$? If yes, calculate the eigenvalues:

$$|1,1\rangle, \quad |-1,-1\rangle, \quad |1,0\rangle + |1,1\rangle, \quad |1,0\rangle + |0,1\rangle, \quad |1,0\rangle - |0,1\rangle,$$
$$|1,-1\rangle + 2|0,0\rangle + |-1,1\rangle, \quad |1,-1\rangle - |-1,1\rangle\,.$$

8. For given values of j_1 and j_2, the number of states in the uncoupled basis $|j_1 m_1, j_2 m_2\rangle$ is $(2j_1 + 1)(2j_2 + 1)$. The number of states in the coupled basis $|JM, j_1, j_2\rangle$ should be the same, since the two bases are linearly related. A partial proof of the Angular Momentum Addition Theorem consists in showing that this is the case provided that the allowed values of j run from $|j_1 - j_2|$ in integer steps to $j_1 + j_2$. You are invited to prove, therefore, that

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1).$$

9. Verify that the states

$$\begin{aligned}
|s=1, m_s=1\rangle &= |\uparrow\uparrow\rangle \\
|s=1, m_s=0\rangle &= \frac{1}{\sqrt{2}}\{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\} \\
|s=1, m_s=-1\rangle &= |\downarrow\downarrow\rangle \\
|s=0, m_s=0\rangle &= \frac{1}{\sqrt{2}}\{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\}
\end{aligned}$$

   of the coupled basis are indeed eigenstates of the operator $\hat{S}^2$ with eigenvalues $s(s+1)\hbar^2$ by using the identity

$$\hat{S}^2 \equiv \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}$$

10. Prove that the parity operator, defined by $\mathcal{P}\psi(x) = \psi(-x)$, is a Hermitean operator. Also prove that the eigenfunctions of $\mathcal{P}$, corresponding to the eigenvalues $+1$ and -1, are orthogonal.

11. Construct the four-dimensional representation of the components of the intrinsic angular momentum $\hat{S}_x, \hat{S}_y, \hat{S}_z$ acting in the space of states with spin $s = 3/2$. Write explicitly the matrix corresponding to the operator $\hat{S}^2$; verify that you obtain the expected result.

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