Quantum Mechanics

Problem Sheet 9

1. A particle moves in the potential

$$V(x) = \infty, \quad |x| > a, \quad V(x) = V_0 \cos(\pi x/2a), \quad |x| \leq a$$

Calculate the energies of the two lowest states to first order in perturbation theory.

2. A particle moves in the potential

$$V(x) = \infty, \quad |x| > a, \quad V(x) = V_0 \sin(\pi x/a), \quad |x| \leq a$$

- show that the first order energy shift is zero;
- obtain an expression for the second order correction to the energy of the ground state.

3. The anharmonic oscillator: a particle of mass m is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \tfrac{1}{2} m\omega^2 \hat{x}^2 + \gamma \hat{x}^4$$

- Assuming that γ is small, use first-order perturbation theory to calculate the ground state energy;
- show more generally that the energy eigenvalues are approximately

$$E_n \simeq (n + \tfrac{1}{2})\hbar\omega + 3\gamma \left(\frac{\hbar}{2m\omega}\right)^2 (2n^2 + 2n + 1)$$

4. The so-called hyperfine interaction in the hydrogen atom between an $\ell = 0$ electron and the spin of the proton which constitutes the nucleus, can be treated by perturbation theory, with

$$\hat{H}' = \tfrac{4}{3} g_p \frac{M_e}{M_p} \alpha^4 M_e c^2 \frac{1}{n^3 \hbar^2} \, \underline{\hat{S}}_e \cdot \underline{\hat{S}}_p \qquad\qquad (g_p = 5.56)$$

where $\underline{\hat{S}}_e$ and $\underline{\hat{S}}_p$ are the operators representing the electron and proton spins, respectively, and α is the fine structure constant.

What are the allowed values of the total spin of the electron-proton system? Use first-order perturbation theory in the coupled basis to calculate the hyperfine splitting of the hydrogen ground state ($n = 1$) and hence show that the wavelength of radiation emmitted in transitions between the triplet and singlet states is approximately 21 cm (famous in radio astronomy as the signature of hydrogen!).

5. Consider a periodic potential in one dimension:

$$V(x) = V(x - a)\,.$$

The operator $\hat{T}_a$ translates the wave function of the system by a:

$$\hat{T}_a \psi(x) = \psi(x + a)\,.$$

Show that

$$\left[\hat{T}_a, \hat{H}\right] = 0\,.$$

Find an expression for the eigenfunction of $\hat{T}_a$ corresponding to the eigenvalue $\lambda = e^{ika}$. These eigenfunctions are called *Bloch waves*.

6. Find the energy levels of an electron in the periodic potential:

$$V(x) = \sum_{n=-\infty}^{+\infty} v_0 \delta(x - na)\,.$$

This is the *Kronig-Penney* model for the energy levels of an electron in a crystal. It describes effectively the band structure of a solid, i.e. the energy values that are allowed for an electron in the crystal. The band structure determines whether a solid is a conductor, a semi-conductor, or an insulator.

This is a long question... See if you can solve it without further hints! Solutions are available.

Have a nice Christmas break! 

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