

Chapter 2

Symmetries of a system

2.1 Classical symmetry

Symmetries play a central role in the study of physical systems. They dictate the choice of the dynamical variables used to characterize the system, induce conservation laws, and set constraints on the evolution of the system. We shall see explicit examples of these features in this course.

Group theory is the natural language to discuss symmetry. In this section we shall develop the mathematical tools needed for most physical applications. We have seen in the previous Chapter that both the solutions of classical equations of motions, and the state vectors of quantum mechanics form linear vector spaces. Symmetries of the physical systems imply the existence of distinctive regular patterns in these vector spaces. The exact structure of these patterns is entirely determined by the group-theoretical properties of the symmetry and *are independent* of other details of the spectrum. As in the case of vector spaces, it is important to work out some examples in detail in order to get familiar with the abstract concepts.

We shall start from the more familiar case of classical mechanics, where we have an intuitive understanding of the concept of symmetry in order to introduce the basic ideas.

Reference frames and transformation laws When studying a physical system, we can think of a *reference frame* as a set of rules which enables the observer to measure physical quantities. For instance, if we are studying a point-like particle, the reference frame is made of a mapping of three-dimensional space, *i.e.* a choice of coordinates, a clock, and a detector, so that at any time t we can identify the position $\mathbf{x}(t)$ of our particle. The same phenomenon can be observed from two different reference frames, which are unambiguously defined as soon as we know the mapping which relates the quantities measured in each reference frame. These mapping is called a *transformation law*.

For instance a system can be rotated according to the transformation law:

$$t \mapsto t, \quad \mathbf{x} \mapsto \mathbf{x}' = R(\theta)\mathbf{x}, \quad (2.1)$$

where R is the matrix that relates the coordinates in the two reference frames.

Invariance under a transformation As we have discussed in Chap. 1 the evolution of a physical system is determined by the equations of motion, *i.e.* by mathematical equations that yield the time dependence of the coordinates of the system in a given reference frame.

Two reference frames are said to be *equivalent* for the study of a given physical system if:

1. *all* physical states that can be realized in one system, can also be realized in the other system;
2. we have the *same* equations of motion in both systems.

Consider a transformation law R , a physical system is said to be *invariant* under R if two reference frames related by this transformation are equivalent. The transformation R is called a *symmetry* of the system.

This concept is best illustrated by a concrete example. Let us consider a point particle, whose evolution defines a trajectory from $x_i(0)$ to $x_i(t)$ in a given reference frame A . If we look at the same particle from the reference frame B , the evolution is given by the trajectory:

$$Rx_i(0) \rightarrow Rx_i(t). \quad (2.2)$$

If the system is invariant under the transformation R , and we start a trajectory from the point $Rx_i(0)$ in A , then the evolution of the system in the reference frame A is the same as the evolution in the reference frame B . And therefore, in the reference frame A , the system evolves from $Rx_i(0)$ to $Rx_i(t)$; *i.e.* the time evolution of the transformed system yields the same results as the transformation of the time-evoluted system.

Let us now consider the constraints on the dynamics of a system that derive from invariance under the transformation R . The equation of motion for a point particle under the action of some force \mathbf{f} is given by:

$$m\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}). \quad (2.3)$$

Starting from an initial state $(\mathbf{x}(0), \dot{\mathbf{x}}(0))$, the equation of motion determines the trajectory of the particle. If we observe the system from a transformed reference frame, the trajectory is given by $R\mathbf{x}(t)$. In order to have invariance, the equation of motion needs to be the same in both reference frames, *i.e.* :

$$m\ddot{\mathbf{x}}' = \mathbf{f}(\mathbf{x}'), \quad (2.4)$$

with the *same* function \mathbf{f} appearing in both Eqs. 2.3, 2.4. Assuming that R does not depend on time, and expressing \mathbf{x} as a function of \mathbf{x}' , we obtain:

$$mR^{-1}\ddot{\mathbf{x}}' = \mathbf{f}(R^{-1}\mathbf{x}'), \quad (2.5)$$

and therefore:

$$R\mathbf{f}(R^{-1}\mathbf{x}') = \mathbf{f}(\mathbf{x}') \quad \text{i.e.} \quad \mathbf{f}(R\mathbf{x}) = R\mathbf{f}(\mathbf{x}). \quad (2.6)$$

Exercise 2.1.1 Prove that a system under the action of a force \mathbf{f} is invariant under translations only if the force itself does not depend on position.

2.2 Quantum symmetry

2.2.1 Introduction

The invariance with respect to a transformation law R in quantum mechanics can be expressed in the following way:

1. the allowed states in both reference frames are the same; they are vectors in the same Hilbert space. The transformation law is a mapping from the Hilbert space of states \mathcal{E} onto itself.
2. starting from the same initial state, we have the same evolution in both reference frames.

If the system is invariant under a change of reference frames, the transformation that implements the change of reference frame is called a *symmetry* of the system.

Let us try to clarify this concept with an explicit example of the action of a transformation on a wave function.

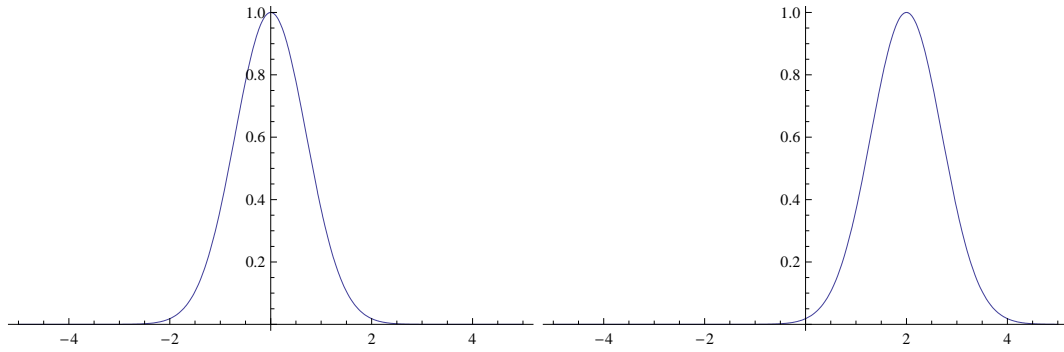


Figure 2.1: The wave function $\psi(x)$ in the left panel describes a state localized around the origin. The wave function $\psi'(x)$ represents the same state translated by two units to the right.

The wave function on the left-hand side in Fig. 2.1 represents a state $|\psi\rangle$ localized around the origin. The plot on the right-hand side in the same Figure shows the wave function for the same system in a translated reference frame.

Given two states $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{E}$ in a given reference frame A , these states are mapped by the transformation R into $|\psi'_1\rangle, |\psi'_2\rangle \in \mathcal{E}$ respectively. These new states describe the system as seen from the new reference frame B . The transition from $|\psi_1\rangle$ to $|\psi_2\rangle$ in A is seen as a transition from $|\psi'_1\rangle$ to $|\psi'_2\rangle$ from the transformed reference frame. If the system is invariant under the transformation, the probability for a transition $|\psi_1\rangle \rightarrow |\psi_2\rangle$ should be the same as the probability for the transition $|\psi'_1\rangle \rightarrow |\psi'_2\rangle$.

Symmetry transformations obey Wigner's theorem:

Theorem 2.2.1 Given a transformation in a Hilbert space \mathcal{E} :

$$|\psi\rangle \rightarrow |\psi'\rangle, \quad (2.7)$$

such that

$$|\langle\psi'_1|\psi'_2\rangle|^2 = |\langle\psi_1|\psi_2\rangle|^2, \forall |\psi_1\rangle, |\psi_2\rangle \in \mathcal{E}, \quad (2.8)$$

it is always possible to choose the complex phase of the quantum states in such a way that the transformation is realized on the vectors as either a unitary or an antiunitary representation.

We shall not prove Wigner's theorem here, even though it plays a central role in describing the symmetry properties of a quantum-mechanical system.

Let us now analyze the implications of the invariance under a given transformation R . As discussed above, a system is invariant under a given transformation only if the time evolution is the same in both reference frames. If we start from a state $|\psi\rangle$ and apply the unitary transformation R , we obtain the state $|\psi'\rangle$. The time evolution of $|\psi\rangle$ is dictated by the Hamiltonian:

$$i\partial_t|\psi\rangle = H|\psi\rangle. \quad (2.9)$$

Therefore the evolution of $|\psi'\rangle$ is obtained as:

$$\begin{aligned} i\partial_t|\psi'\rangle &= i\partial_t R|\psi\rangle = RH|\psi\rangle \\ &= RH(R^\dagger R)|\psi\rangle = RHR^\dagger|\psi'\rangle, \end{aligned} \quad (2.10)$$

i.e. $H' = RHR^\dagger$. Since the system is invariant we must have $H' = H$, and hence R is symmetry transformation of the system only if:

$$[R, H] = 0. \quad (2.11)$$

2.2.2 Particle on a one-dimensional lattice

The main features of symmetry in quantum mechanics can be seen in a very simple example.

Let us consider a point-particle (*e.g.* an electron) in a one-dimensional lattice with lattice spacing b . The dynamics of the system is specified by the Hamiltonian:

$$H = p^2/(2m) + V(x), \quad (2.12)$$

where m is the mass of the electron, p its momentum, and $V(x)$ is the periodic potential due to the lattice in which the electron is evolving. The potential $V(x)$ is a periodic function:

$$V(x + nb) = V(x), \forall n \in \mathbb{Z}. \quad (2.13)$$

Exercise 2.2.1 Sketch a one-dimensional periodic potential with period b .

The transformation law:

$$x \mapsto x' = x + nb, \quad (2.14)$$

is clearly a symmetry of the system, since it leaves the Hamiltonian unchanged. The electron will appear to behave in exactly the same manner to two observers that are related by one of these translations.

Exercise 2.2.2 Check that the classical Hamiltonian is invariant under this transformation. You need to verify that both the kinetic and the potential term are left unchanged.

We shall now translate this physical property into mathematical language. The state of the system is described a state vector $|\psi\rangle$ in the Hilbert space \mathcal{E} . The transformed state after a translation by n steps of size b will be denoted $|\psi'\rangle$. We shall define $T(n)$ as the operator that relates $|\psi\rangle$ to $|\psi'\rangle$. $T(n)$ is an operator mapping \mathcal{E} onto itself:

$$T(n) : \mathcal{E} \rightarrow \mathcal{E}, \quad |\psi\rangle \mapsto |\psi'\rangle = T(n)|\psi\rangle. \quad (2.15)$$

According to Wigner's theorem, the operators $\{T(n)\}$ form a unitary representation of the symmetry of the Hamiltonian.

Physical observables in quantum mechanics are represented by Hermitean operators. Under the transformation $T(n)$, each operator is mapped into:

$$O \mapsto O' = T(n)OT(n)^{-1}. \quad (2.16)$$

Exercise 2.2.3 Show that the operator O' is Hermitean, and therefore also corresponds to an observable.

Hint: the fact that $T(n)$ is unitary plays an important role.

Let $|x\rangle$ be an eigenvector of the position operator X , then

$$T(n)|x\rangle = |x + nb\rangle \quad (2.17)$$

Let us now consider the action of the operator $V(X)$ on the states $|x\rangle$:

$$V(X)|x\rangle = V(x)|x\rangle, \quad (2.18)$$

where $V(X)$ is an operator acting on kets, while $V(x)$ is a number, *i.e.* the actual value of the potential at the position x . The operator $V(X)$ can be replaced by its eigenvalue because $|x\rangle$ is an eigenvector of the operator X .

Starting from Eq. 2.18 we can easily obtain the following result:

$$T(n)V(X)T(n)^{-1}|x\rangle = T(n)V(X)|x - nb\rangle = T(n)V(x - nb)|x - nb\rangle \quad (2.19)$$

$$= V(x - nb)|x\rangle, \quad (2.20)$$

since the operators $\{|x\rangle\}$ form a basis in \mathcal{E} , we can deduce the following equality *between operators*:

$$T(n)V(X)T(n)^{-1} = V(X - nb). \quad (2.21)$$

Exercise 2.2.4 Using the above results, prove that the quantum Hamiltonian (*i.e.* the operator H acting on the vector space of quantum states) is invariant under $T(n)$: $[T(n), H]$.

For the discrete translations one can easily visualize that the following properties hold:

1. $T(n)T(m) = T(n+m)$, the successive application of two translations yields another translation;
2. $T(0) = E$, where E indicates the *identity* operator;
3. $T(-n) = T(n)^{-1}$, *i.e.* each translation has an inverse, which is the translation in the opposite direction.

These properties characterize the set of translations as a *group*. The rest of this Chapter is devoted to a formal definition of groups, and to the study of their properties.

2.3 Symmetry groups in physics

Group theory has important consequences in the study of quantum physical systems. We list here some important properties.

1. Since the Hamiltonian commutes with the symmetry transformations, there are basis which diagonalize H and some group transformations simultaneously.
2. the group-theoretical properties do not depend on the details of the physical system and can be found by general mathematical methods.
3. the physical interpretation of the mathematical results leads to valuable informations on the states of the system.

We end this Section by listing here a number of important examples of symmetries that are commonly used in physics.

1. translations in space: $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$; momentum conservation follows from the invariance under space translations, see *e.g.* [LL76, Gol80].
2. translations in time: $t \mapsto t + t_0$; in this case, invariance under time translations implies energy conservation, see *e.g.* [LL76, Gol80].
3. rotations in three-dimensional space; this symmetry is associated with conservation of angular momentum, see *e.g.* [LL76, Gol80].
4. Lorentz transformations; they combine space and time transformations together and are the basis of Einstein's theory of special relativity.
5. Parity transformation: $\mathbf{x} \mapsto -\mathbf{x}$; most interactions are symmetry under this transformation, with the notable exception of weak interactions in particle physics. [Sak64]

6. Time reversal: $t \mapsto -t$; this symmetry is respected by almost all physical systems, with only tiny violations. [Sak64]
7. permutation symmetry; systems with identical particles are invariant under the exchange of these particles. This symmetry has important physical consequences.
8. gauge invariance; electromagnetism is invariant under gauge transformations. The generalization of gauge symmetry to other groups is one of the guiding principles in constructing the Standard Model of particle physics.
9. internal symmetries of strong interactions; this type of symmetry plays a crucial role in identifying the spectrum of elementary particles and their properties.

2.4 Group Theory

We shall now build a general abstract framework that will be useful to discuss all the examples of symmetry that appear in Nature. The mathematical tools required are known as *group theory*. There are many good references for those who want to study group theory in more detail. Most of the material presented here can be found in [Tun85] and [Arm88].

2.4.1 Basic definitions

Let us start from the formal definition of a *group*.

Definition 2.4.1 Let us consider the pair (G, \cdot) , where G is a set, and the dot denotes a binary operation, $G \times G \rightarrow G$.

(G, \cdot) is called a *group* if and only if:

1. $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$; the product rule is *associative*.
2. $\exists e \in G$, such that $\forall a \in G, e \cdot a = a \cdot e = a$; there is an *identity* element in G .
3. $\forall a \in G, \exists a^{-1} \in G$, such that $a \cdot a^{-1} = a^{-1} \cdot a = e$; every element of G has an *inverse* in G .

Note that the group G must be closed under the operation \cdot , *i.e.* the product of two elements of G *must* yield an element of G . This property needs to be checked explicitly in order to establish whether a given pair (G, \cdot) defines a group or not. These properties are explicitly verified by the discrete translations that we considered in the previous Section.

Example We can start with an example of a very simple group. Let us consider the set $G = \{0, 1\}$, and define the binary operation $a \cdot b = (a + b) \bmod 2$, *i.e.* the ordinary sum of a and b modulo 2.

We can encode the results of the binar operation in a table:

\cdot	0	1
0	0	1
1	1	0

Such a table is called a *multiplication table*. Note that \cdot really defines a mapping $G \times G \rightarrow G$; to each pair of elements of G , \cdot associates an element of G . The identity element in G is 0, and each element has its inverse in G - you can read from the multiplication table that, in this example, each element is its own inverse. Clearly the first line and the first column in the table are trivial, and can be omitted. The multiplication table is more conveniently written as:

	0	1
1	1	0

Example A slightly more abstract example is the following. Consider a set of n elements: $G = \{e, a, a^2, \dots, a^{n-1}\}$, where a^j indicates the product $a \cdot a \dots a$ (j times), e is the identity, and we require $a^n = e$. This group is called the *cyclic group* of order n , and denoted Z_n .

Exercise 2.4.1 Check that G is a group by identifying the product of two arbitrary elements of G , and by identifying the inverse of each element.

Definition 2.4.2 A group (G, \cdot) is called *Abelian* iff:

$$\forall a, b \in G, \quad a \cdot b = b \cdot a, \quad (2.22)$$

or equivalently

$$a \cdot b \cdot a^{-1} \cdot b^{-1} = e. \quad (2.23)$$

The product $a \cdot b \cdot a^{-1} \cdot b^{-1}$ is called the *group commutator* of a and b .

Exercise 2.4.2 Check that $(\mathbf{Z}, +)$ is a group. Here $+$ indicates the usual sum of integer numbers. Is it an Abelian group?

Definition 2.4.3 Two elements $h, h' \in G$ are called *conjugate elements*, and denoted $h \sim h'$, iff:

$$\exists g \in G, \quad h' = g \cdot h \cdot g^{-1}. \quad (2.24)$$

Exercise 2.4.3 If the group Abelian, show that

$$h \sim h' \iff h = h'. \quad (2.25)$$

Note that “ \sim ” defines an *equivalence relation*, i.e.

1. $h \sim h$ (reflexive);
2. $h \sim h' \iff h' \sim h$ (symmetric);
3. $h \sim h', h' \sim h'' \implies h \sim h''$ (transitive);

Exercise 2.4.4 Check explicitly that the relation \sim defined in Eq. 2.24 is an equivalence relation.

Definition 2.4.4 The *order* $|G|$ of a group G is the number of elements of the group (if it is finite).

Example The order of the cyclic group Z_k is k .

We shall now define the order of group element, not to be confused with the order of the group.

Definition 2.4.5 Let $a \in G$, if $\exists m \in \mathbb{N}$, $a^m = e$, let n be the smallest of such m 's, then a is of *order* n . Otherwise, if all powers of a are distinct, a is of *infinite order*.

Note that in a finite group, all elements must have order $< |G|$.

Exercise 2.4.5 Prove that the order of *any* element g of the cyclic group Z_n is a divisor of n , i.e. let $g \in G$ be of order m , show that $n/m = k$, with k integer.

Exercise 2.4.6 If $a^2 = b^2 = (ab)^2 = e$ (identity) show that $ab = ba$. Deduce that a group in which every element, except the identity, of order 2 is Abelian.

Multiplication table for a finite group. A finite group is uniquely defined by a complete description of its product law: $(g_1, g_2) \mapsto g_1 \cdot g_2$. As we have seen in previous examples, such description can be encoded in a *multiplication table*.

Example Let us consider the two possible multiplication tables for a finite group of order 4.

1. $|G| = 4$, all elements of order 4; this is the cyclic group $Z_4 = \{e, a, a^2, a^3\} \equiv \{e, a, b, c\}$. The multiplication table is given by:

e	a	b	c
a	b	c	e
b	c	e	a
c	e	a	b

2. $|G| = 4$, all elements of order 2; this is called the “Vier gruppe”, and denoted V_4 . The multiplication table in this case is:

e	a	b	c
a	e	c	b
b	c	e	a
c	b	a	e

Exercise 2.4.7 Check that the multiplication table for V_4 is the only multiplication table compatible with the hypotheses that all elements are distinct and of order 2.

These are the only two multiplication tables allowed for a finite group of order 4.

Many properties of finite groups can be derived from the following theorem.

Theorem 2.4.1 The arrangement of elements in a column (row) of a group multiplication table is different from that in every other column (row).

(*Rearrangement theorem*)

The proof of the theorem is quite simple, we report it here as an example for the interested reader. Suppose that an element d occurs twice in a column, then

$$\exists b, c \in G, \text{ distinct, such that } ba = d, ca = d; \quad (2.26)$$

but this last statement implies

$$b = da^{-1} = c, \quad (2.27)$$

which contradicts the hypothesis that $b \neq c$.

Hence there is no element that can appear more than once in any column, *i.e.* each element appears exactly one time. Thus each column contains a unique arrangement of the group elements.

The following property is an obvious consequence of the rearrangement theorem.

Theorem 2.4.2 If f is any function of the group elements, then

$$\sum_{a \in G} f(a) = \sum_{a \in G} f(ga), \quad \forall g \in G. \quad (2.28)$$

Dihedral group A slightly more complex example is provided by the *dihedral group* D_3 , *i.e.* the group of symmetry operations of the equilateral triangle. There are six elements in this group, corresponding to the six geometrical transformation that leave the equilateral triangle unchanged. They can be namely identified as:

1. the identity e ;
2. the anticlockwise rotation by $2\pi/3$, which we call a ;
3. the anticlockwise rotation by $4\pi/3$, which we call b , note that $a^2 = b$, and $ab = e$;
4. the three reflexions with respect to the axes of the triangle, which we denote c , d , and f . The three axes are shown explicitly in Fig. 2.2. (note that $c^2 = d^2 = f^2 = e$)

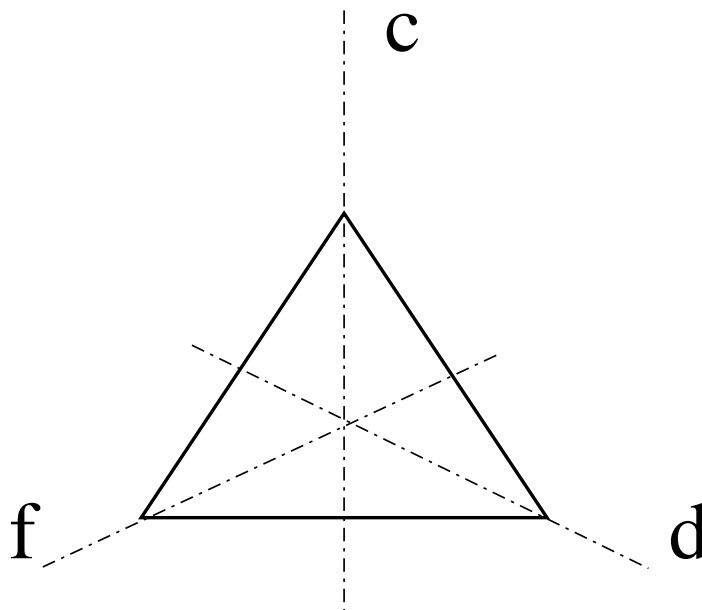
Exercise 2.4.8 Show that the multiplication table for D_3 is:

e	a	b	c	d	f
a	b	e	f	c	d
b	e	a	d	f	c
c	d	f	e	a	b
d	f	c	b	e	a
f	c	d	a	b	e

Generators and rank Let $G = \{g_1, \dots, g_n\}$ be a finite group of order n , with $g_1 = e$, we can define the following two concepts.

Definition 2.4.6 A subset of group elements $\{t_1, \dots, t_s\}$ such that *all* elements of G can be written as a product of t_i 's is said to *generate* the group G . The smallest such set is the set of *generators* of G .

Definition 2.4.7 The number of generators is called the *rank* of G .

Figure 2.2: Axes of the equilateral triangle which define the reflexions c, d, f .**Examples**

1. Z_k has rank 1.
2. D_3 is generated by a and c , and therefore has rank 2.

Subgroups A subgroup is a subset of a group whose elements satisfy the properties that define a group.

Definition 2.4.8 A subset $H \subset G$ is a *subgroup* of G if and only if it is a group, *i.e.*

1. $\forall h_1, h_2 \in H, h_1 h_2 \in H$ (closure of H);
2. $\forall h \in H, h^{-1} \in H$ (existence of the inverse in H).

Note that the inverse of h is guaranteed to exist as an element of G , since G is a group and $H \subset G$. The non-trivial property that needs to be checked is the fact that the inverse is an element of H .

Example Consider the set of all even integers $(2\mathbb{Z}, +)$, with the product law being the usual addition of integers. You can easily verify that it is a subgroup of the group $(\mathbb{Z}, +)$ of integers. What about the set of odd integers?

Example Using the notation introduced above for the group D_3 , you can easily check that the subset $H = \{e, a, b\}$ is a subgroup.

2.4.2 Permutations

In order to illustrate some of the concepts introduced in the previous section, let us describe in some detail the *group of permutations*. The permutation of a set of symbols is an intuitive and familiar idea. In a more mathematical language, we can define a permutation of an arbitrary set X as a one-to-one correspondence from X into X . It is straightforward to check that the set of permutations has the structure of a group. If π_1 and π_2 are two permutations, the composite function $\pi_1\pi_2$ is also a one-to-one correspondence from X to X , and hence is a permutation. There is a special permutation ε leaving all elements unchanged which plays the role of the identity, and each permutation can be inverted (because it is a one-to-one correspondence). When the set X is finite, and contains n elements, the group of permutations is denoted S_n , and called the *symmetric group* of degree n .

Exercise 2.4.9 Check that the order of S_n is $n!$.

Let us concentrate on the symmetric group S_3 . The six elements of S_3 are denoted:

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}. \quad (2.29)$$

Remember that the elements of S_3 are permutations of the set of three elements $\{1, 2, 3\}$. Each permutation is identified in this notation by writing the image of each element vertically underneath it inside the parenthesis. So for instance the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad (2.30)$$

sends $1 \mapsto 1$, $2 \mapsto 3$, and $3 \mapsto 2$.

In order to avoid ambiguities, remember that the notation $\pi_1\pi_2$ means that you *first* apply π_2 and *then* π_1 .

Exercise 2.4.10 Show that

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad (2.31)$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}. \quad (2.32)$$

Deduce that S_3 is not Abelian.

A better notation The notation introduced above is very clear, but becomes fairly awkward as the degree of the symmetric group is increased. For example, an element of S_6 is usually written as:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 6 & 1 & 2 \end{pmatrix}. \quad (2.33)$$

Exercise 2.4.11 Make sure that you understand the action of the above permutation on the set of integers $\{1, 2, 3, 4, 5, 6\}$.

The same information can be encoded by writing: $\pi = (15)(246)$, which indicates that one should perform a permutation of 1 and 5, and then send 2 to 4, 4 to 6, and 6 to 2. The element 3 is never mentioned in this notation since it is left unchanged.

The general prescription for describing a permutation in this notation is the following: Open a pair of brackets, write down the smallest integer which is moved by permutation, followed by its image, followed by the image of the image, and so on, until eventually you come full circle to the starting point, in which case you close the bracket. Now open a new set of brackets, and perform the same operation as before, starting with the smallest integer which has not been mentioned so far, and then list all the subsequent images. Iterate until all integers are exhausted.

Exercise 2.4.12 Check that the six elements of S_3 listed above can be written as:

$$\varepsilon, (123), (132), (12), (13), (23). \quad (2.34)$$

The permutation inside a single bracket is called a *cyclic permutation*. The number of integers in the cyclic permutation is called the *length* of the permutation, and a cyclic permutation of length k is often referred to as a *k-cycle*.

2.4.3 Maps between groups

We have seen several examples of groups; they all correspond to explicit realizations of abstract concepts. In a concrete realization the elements of the group (and the product law) become explicit objects (numbers, matrices, geometrical transformations) that we can manipulate according to well-defined rules.

In this section, we want to define the mathematical tools that are used to compare different groups.

Definition 2.4.9 Let (G, \cdot) and (G', \times) be two groups, a map $\phi : G \rightarrow G'$ is called a *homomorphism* if and only if:

$$\forall g_1, g_2 \in G, \quad \phi(g_1 \cdot g_2) = \phi(g_1) \times \phi(g_2). \quad (2.35)$$

Note that when we write $\phi(g_1 \cdot g_2)$ the “ \cdot ” acts in G ; on the other hand in the expression $\phi(g_1) \times \phi(g_2)$ the operation “ \times ” on elements of G' .

A homomorphism $D : G \rightarrow \text{GL}(V)$, where $\text{GL}(V)$ is the set of linear transformations acting on a vector space V is called a *linear representation* of G . We shall discuss representations in great detail later.

Definition 2.4.10 An homomorphism $\phi : G \rightarrow G'$ which is also invertible is called an *isomorphism* (one-to-one correspondence).

Two groups G, G' such that there is an isomorphism $\phi : G \rightarrow G'$ are called *isomorphic*. They are different realizations of the same abstract group structure.

An isomorphism $D : G \rightarrow \text{GL}(V)$ defines a *faithful representation* of G .

Finite groups can be classified by mapping them to subgroups of the groups of permutations. This result is summarized in the following theorem.

Theorem 2.4.3 Any group G of finite order $n = |G|$ is isomorphic to a subgroup of the symmetric group S_n .
(Cayley's theorem)

We can build the isomorphism $\phi : G \rightarrow H \subset S_n$ explicitly. The crucial observation is that each row of the multiplication table of a finite group defines a permutation of the elements of the group. Hence to any given element of $g \in G$, we can associate a permutation $\pi_g \in S_n$:

$$\pi_g = \begin{pmatrix} g_1 & \cdots & g_n \\ gg_1 & \cdots & gg_n \end{pmatrix}. \quad (2.36)$$

It is easy to prove that the mapping defined above respect the product law of the two groups, *i.e.* $\pi_a \pi_b = \pi_{ab}, \forall a, b \in G$.

A simple corollary of Cayley's theorem is that the number of groups of finite order n is finite, since the number of subgroups of S_n is finite.

Definition 2.4.11 Permutations π_g associated to an element $g \in G$ can be read directly from a multiplication table. They are called *regular permutations*.

Note that regular permutations do not leave any symbol unchanged.

2.4.4 Cosets and Lagrange theorem

Let H be a subgroup of G , $|H| = m$, $H = \{h_1 = e, h_2, \dots, h_m\}$, and let $a_1 \in G, a_1 \notin H$, we can define the *left coset* of H :

Definition 2.4.12 the set $a_1 H = \{a_1 h_1 = a_1, a_1 h_2, \dots, a_1 h_m\}$ is called a left coset of H .

Note that we can define a coset for each element of G . If $a_1 \in H$ then, due to the rearrangement theorem, $a_1 H = H$. Right cosets are defined in a completely analogous way, by multiplying the elements of H to the right. You should also realize that cosets are *not* subgroups. The simplest way to see that this is indeed the case is to notice that cosets do not contain the identity element e .

Cosets satisfy two simple properties:

1. $a_1 h_i \neq a_1 h_j$, if $i \neq j$;
2. $a_1 h_i \notin H$, $\forall i = 1, \dots, m$.

In order to get some practice with manipulations in group theory, let us prove the second property. Let us assume that $a_1 h_i \in H$, then there must be an element in H , denoted h_j such that $a_1 h_i = h_j$. Then:

$$a_1 h_i = h_j \quad (2.37)$$

$$\Rightarrow a_1 = h_j h_i^{-1} \quad (2.38)$$

$$\Rightarrow a_1 \in H, \quad (2.39)$$

which contradicts the initial assumption $a_1 \notin H$.

Theorem 2.4.4 Two left (or right) cosets of H in G are either identical or disjoint.

In order to prove the theorem, consider two cosets a_1H and a_2H , and assume they have *one* element in common g . Then there exist $h_1, h_2 \in H$, such that $g = a_1h_1 = a_2h_2$. From the previous equality we deduce: $a_2^{-1}a_1 = h_2h_1^{-1} \in H$. Therefore $a_2^{-1}a_1H = H$, i.e. $a_1H = a_2H$.

The fact that cosets are always disjoint, i.e. they do not have elements in common unless they coincide exactly, allows us to prove the following:

Theorem 2.4.5 if a group H of order h is a subgroup of a group G of order g , then $g = lh$ for some integer l . The integer l is called the *index* of H in G .

(Lagrange's theorem)

Proof: let us take $g_1 \in G, g_1 \notin H$, and construct g_1H . Not there are two possibilities: either H and g_1H exhaust all elements in G , or there exist an element $g_2 \in G, g_2 \notin H, g_1H$. In the former case the theorem $l = 2$ and we have proved the theorem. In the latter case we can construct another coset g_2H , which contains h elements, disjoint from the other two. We can iterate the procedure until all elements in G have been used. We end up with l cosets g_1H, \dots, g_lH which all have h elements. Hence $g = lh$.

Note that the period of a group element, defined as the set $\{a, a^2, \dots, a^k = e\}$, is a subgroup of G , of order k . As a consequence of Lagrange's theorem, the order of each element $a \in G$ must be a factor of the order of the group G .

Exercise 2.4.13 The group D_3 is of order 6. The subgroups *must* be of order 2, or 3. Find the subgroups of D_3 and check that this property is verified. Check also that the order of the rotations is 3, while the order of the reflections is 2.

(Hint: make sure you remember the difference between the order of a group (or subgroup) and the order of an element. In the first part of the question you have to find out subgroups of D_3 and check their order. In the second part of the question you have to consider the order of the elements.)

2.4.5 Conjugacy classes

Conjugate elements have been introduced in Def. 2.24. Because of transitivity of the equivalence relation, we can split the elements of G into sets subsets such that all elements in a given subset are conjugate to each others. Such sets are called *conjugacy classes*, or simply *classes* of G , and are labelled by one of their members:

Definition 2.4.13 the conjugacy class $[g]$ is defined as:

$$[g] = \{g' \in G, g' \sim g\} . \quad (2.40)$$

Note that the classes are *not* subgroups of G , and that the identity element in G always forms a class by itself, i.e. the only element that is conjugated to e is e itself, $[e] = \{e\}$.

Exercise 2.4.14 Using the multiplication table, find the conjugacy classes in D_3 .

In a similar manner we can define the concept of a *conjugate subgroup*:

Definition 2.4.14 Let us consider $H \in G$, $g \in G$, and let H be a subgroup, the set $H_g = ghg^{-1}, \forall h \in H$ is a subgroup called the conjugate subgroup to H .

The notion of conjugate subgroup leads to the following definition:

Definition 2.4.15 $H \subset G$ is called an *invariant* subgroup if and only if: $\forall g \in G, H_g = H$, i.e. if all the subgroups conjugate to H actually coincide with H .

If H is invariant then: $gH = Hg, \forall g \in G$, i.e. the left and right cosets of H coincide.

If H is invariant we can define the *factor group*.

Definition 2.4.16 Let $H \subset G$, $gH = Hg$, and let us denote by G/H the set of all cosets of H . Then G/H is a group, with the respect to the product law:

$$(g_1H) \cdot (g_2H) = (g_1g_2)H. \quad (2.41)$$

Note that on the LHS of the above equation we are defining a binary law between cosets. The RHS makes use of the product law in the original group G . The set of cosets has a group structure *only* if the subgroup H is invariant.

Finally we can define:

Definition 2.4.17 the *direct product* of two groups H and K as the set:

$$H \otimes K = \{(h, k), h \in H, k \in K\}. \quad (2.42)$$

$H \otimes K$ is a group with the product law:

$$(h, k) \cdot (h', k') = (hh', kk'). \quad (2.43)$$

Note that $|H \otimes K| = |H||K|$.

Exercise 2.4.15 Prove that H and K are isomorphic to two invariant subgroups of $G = H \otimes K$. Prove that $G/H = K$ and $G/K = H$.

Example Consider the product group $Z_2 \otimes Z_2 = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$. It clearly is a finite group of order 4. We have seen that there only two possible multiplication tables for a finite group of order 4. Construct the multiplication table for $Z_2 \otimes Z_2$ according to the rule in Eq. 2.43, and show that it is isomorphic to V_4 .

2.5 Problems

2.5.1 Examples of groups

Show that the following sets are groups under the given laws of composition. Specify the identity element and identify the inverse of each element.

1. the set of all rational numbers ¹ under *addition*;
2. the set of all non-zero rational numbers under *multiplication*;
3. the set of all complex numbers under *addition*;
4. the set of all non-zero complex numbers under *multiplication*;
5. the set of all real orthogonal $n \times n$ matrices under *matrix multiplication*;
6. the set of all real orthogonal $n \times n$ matrices with determinant = 1 under *matrix multiplication*;
7. the set of all unitary $n \times n$ matrices under *matrix multiplication*;
8. the set of all unitary $n \times n$ matrices with determinant = 1 under *matrix multiplication*;
9. the set of functions:

$$a(x) = \frac{1}{x}, \quad b(x) = 1 - x, \quad c(x) = \frac{x}{x-1}, \quad f(x) = x, \quad g(x) = \frac{1}{1-x}, \quad h(x) = \frac{x-1}{x}$$

with composition $(f_1 f_2)(x) = f_1(f_2(x))$.

2.5.2 Sets that are NOT groups

Show that the following sets are not groups under the given laws of composition. Which of the group axioms do they fail to satisfy?

1. the set of all real numbers under *multiplication*;
2. the set of all non-negative real numbers under *addition*;
3. the set of all odd integers under *multiplication*;
4. the set $(1, 2, \dots, p-1)$ of $p-1$ integers under *multiplication modulo p* , where p is not prime.
5. the set of vectors \underline{v} under the vector product $\underline{v} \times \underline{w}$.

2.5.3 Centre of a group

(a) Show that the set of elements of a group G which commute with *every* element of G is a subgroup of G (called the *centre* of G).

(b) Show similarly that the set of elements of a group G which commute with a given element a of G is a subgroup of G (called the *normaliser* of a in G).

¹A rational number is expressible as the ratio of two integers

2.5.4 Quaternion group

A group of order 8 is generated by two elements a and b of order 4, such that $ab = ba^{-1}$ and $b^2 = a^2$. The eight elements are thus

$$e, \quad a, \quad b, \quad c \equiv ab, \quad \bar{e} \equiv a^2 = b^2, \quad \bar{a} \equiv a^3, \quad \bar{b} \equiv b^3, \quad \bar{c} \equiv c^3.$$

Construct the multiplication table of the group $\{e, a, b, c, \bar{e}, \bar{a}, \bar{b}, \bar{c}\}$.

Verify that this group, known as the *quaternion group*, may be realised by a group of 2×2 matrices in which a, b, c correspond to $-i\sigma_1, -i\sigma_2, -i\sigma_3$, where the three matrices σ_i are the Pauli spin matrices.

Write down the matrices which correspond to the other elements.

2.5.5 SU(2)

SU(2) is the set of 2×2 complex matrices, unitary, and with $\det = 1$.

1. Make sure you understand that SU(2) is a group under matrix multiplication.
2. Show that any element g of SU(2) can be parameterized as:

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with } a, b \in \mathbb{C}, \text{ and } |a|^2 + |b|^2 = 1 \quad (2.44)$$

3. Writing $a = u_0 + iu_3$, $b = u_2 + iu_1$, show that:

$$g = u_0 + i \sum_{k=1}^3 u_k \sigma_k \quad (2.45)$$

where u_0, u_1, u_2, u_3 are real and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices.

4. Rewrite the constraint $|a|^2 + |b|^2 = 1$ in terms of u_0, u_1, u_2, u_3 .
Deduce that: $\dim[SU(2)] = 3$ and that each element of SU(2) is parameterized by a point on the 3-dim sphere S_3 .
 $S_3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R} : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$.

2.5.6 Using the rearrangement theorem

Consider the finite group $G = \{e, a, b, c\}$, $|G| = 4$.

Suppose that all elements, $\neq e$, are of order 2.

Show that the multiplication table is uniquely determined and write it down explicitly.

(Hint: use the rearrangement theorem)

2.5.7 Classes of a group

Show that an element of a group G constitutes a class by itself if, and only if, it commutes with all elements of G . Hence show that, in an Abelian group, every element is a class.

2.5.8 Multiples of integers

Consider $n\mathbb{Z} = \{\dots, -2n, -n, 0, n, 2n, \dots\}$.

1. Show that $(n\mathbb{Z}, +)$ is a group.
2. Show that $n\mathbb{Z} \subset \mathbb{Z}$ and invariant.
3. Show that the factor group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the cyclic group Z_n .

2.5.9 Factor group

Consider the group D_3 . Show that

1. $H = \{e, a, b\}$ is a subgroup of D_3 of index 2;
2. H is invariant in G ;
3. the factor group $K = G/H$ has two elements, H and H_2 ;
4. K is isomorphic to Z_2 .

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