Lecture 14

Spin

14.1 Introduction

Using the commutation relations of the components of the angular momentum we have found that the allowed eigenvalues for $\hat{J}^2$ are $\hbar^2 j(j+1)$, where $j = 0, \frac{1}{2}, 1\frac{3}{2}, \ldots$. For each value of j, the eigenvalues of J_z are $\hbar m$, with $m = -j, -j+1, \ldots, j-1, j$.

Comparing with the solutions of the eigensystem discussed in lecture 8, we see that we have found more solutions than there are in Eq. (8.21). Eq. (8.21) is a partial differential equation in θ and ϕ. The solutions of this equation are the sperical harmonics $Y_\ell^m(\theta, \phi)$, Eq. (8.22). We can see from the explicit expression for the spherical harmonics that the ϕ dependence is simply:

$$Y_\ell^m(\theta, \phi) \propto \exp(im\phi)\,, \tag{14.1}$$

as expected, since the spherical harmonics are *also* eigenfunctions of $L_z = -i\hbar\frac{\partial}{\partial\phi}$. Since we required that wave functions must be single-valued, the sperical harmonics must be periodic in ϕ with period 2π:

$$Y_\ell^m(\theta, \phi) = Y_\ell^m(\theta, \phi + 2\pi)\,. \tag{14.2}$$

Eq. (14.2) requires m to be integer, and hence j must also be an integer.

In order to understand the physical meaning of the solutions with half-integer j, let us investigate their properties in more detail.

14.2 Matrix representation

For $j = \frac{1}{2}$, we can compute the matrix elements: $\langle\frac{1}{2}, m'|\hat{J}_i|\frac{1}{2}, m\rangle$; the possible values for m' and m are: $m' = \frac{1}{2}$ or $-\frac{1}{2}$ and $m = \frac{1}{2}$ or $-\frac{1}{2}$. If we choose the convention that the row and column labels start with the largest value of the magnetic quantum number and decrease, so that the first row (column) is labelled by $m'(m) = \frac{1}{2}$ and the second row (column) is labelled by $m'(m) = -\frac{1}{2}$, we find the following 2×2 matrix:

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We say that this matrix *represents* the operator $\hat{J}_z$ in the $j = \frac{1}{2}$ multiplet.

The only non-zero element for the matrix representing $\hat{J}_+$ is when $m' = \frac{1}{2}$ and $m = -\frac{1}{2}$, for which

$$c_+ = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) + \tfrac{1}{2}(\tfrac{1}{2})} = 1\,,$$

and hence the matrix is

$$\hat{J}_+ \longrightarrow \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

whilst the only non-zero element for the matrix representing $\hat{J}_-$ is when $m' = -\frac{1}{2}$ and $m = \frac{1}{2}$, for which

$$c_- = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) + \tfrac{1}{2}(\tfrac{1}{2})} = 1\,,$$

also, giving

$$\hat{J}_- \longrightarrow \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

From these two matrices it is easy to construct the matrices representing $\hat{J}_x$ and $\hat{J}_y$, since

$$\begin{aligned} \hat{J}_x &= \frac{1}{2}(\hat{J}_+ + \hat{J}_-)\,, \\ \hat{J}_y &= \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)\,. \end{aligned}$$

Thus

$$\hat{J}_x \longrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{J}_y \longrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

You can readily verify that these 2×2 matrices satisfy the angular momentum commutation relations from which we started. We say, therefore, that they *provide a matrix representation* of the angular momentum operators.

The set of three numerical 2×2 matrices which appear above in the matrix representations of $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$ are known as the *Pauli spin matrices* and are usually denoted as follows:

$$\boxed{\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Collectively, we can write

$$\underline{\hat{J}} \longrightarrow \frac{1}{2}\hbar\,\underline{\sigma}\,,$$

meaning $\hat{J}_x \longrightarrow \frac{1}{2}\hbar\,\sigma_x$, etc. Often we will just write $=$ instead of $\longrightarrow$, but you should remember that this is just one possible choice for representing the operators.

The Pauli matrices have the following property, which you can easily verify

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1\,,$$

where 1 denotes the unit 2×2 matrix.

14.3 Eigenvectors

It is trivial to show that the matrix σ_z has eigenvectors which are just two-component column matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so that the eigenvalue equations for $\hat{J}_z$ are

$$\frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and we see that $\hat{J}_z$ has eigenvalues $\pm\frac{1}{2}\hbar$ as it should for a system with $j = \frac{1}{2}$.

Furthermore, if we construct the matrix representing $\hat{J}^2$, we see that it has these same two column matrices as eigenvectors with a common eigenvalue $j(j+1)\hbar^2 \equiv \frac{3}{4}\hbar^2$:

$$\hat{J}^2 \equiv \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = \frac{1}{4}\hbar^2\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2\right] = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can thus identify the two column matrices with the two simultaneous eigenstates of the operators $\hat{J}^2$ and $\hat{J}_z$:

$$|j = \tfrac{1}{2}, m = \tfrac{1}{2}\rangle \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |j = \tfrac{1}{2}, m = -\tfrac{1}{2}\rangle \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can also see that an arbitrary state $|\psi\rangle$ with $j = \frac{1}{2}$ may be represented as a linear combination of these two states since they span the two-dimensional space of 2-component column matrices:

$$|\psi\rangle = \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} c_m\, |j = \frac{1}{2}, m\rangle$$

is represented by

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = c_{\frac{1}{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_{-\frac{1}{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

14.4 Scalar Products

The rule for scalar products is different when using a matrix representation; it doesn't involve any integration. The Dirac kets we have seen are represented by column matrices of rank $(2j+1)$; the corresponding conjugates, or Dirac bras, are represented by row matrices of the same rank. The rule is that if

$$|\psi\rangle \longrightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \text{then} \qquad \langle\psi| \longrightarrow \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix} .$$

The scalar product of two states $|\psi\rangle$ and $|\phi\rangle$ is then defined to be

$$\boxed{\langle\phi|\psi\rangle \equiv \begin{pmatrix} \phi_1^* & \phi_2^* \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \phi_1^*\psi_1 + \phi_2^*\psi_2} \, .$$

Thus, for example, for a normalised state

$$\langle\psi|\psi\rangle = \psi_1^*\psi_1 + \psi_2^*\psi_2 = |\psi_1|^2 + |\psi_2|^2 = 1$$

The orthonormality property of the eigenvectors is also obvious:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1, \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1, \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \, ,$$

and these may be used to project out the coefficients c_m in the expansion of the arbitrary state $|\psi\rangle$.

14.5 Eigenvectors of $\hat{J}_x$

So far we have concentrated on the eigenvalues and eigenstates of $\hat{J}_z$, but what of the other Cartesian components of angular momentum ? It is clear that, since there is nothing special about the z direction, we should also expect that measuring say the x component of the angular momentum for a system with $j = \frac{1}{2}$ can only yield either $\pm\frac{1}{2}\hbar$. Let us verify this. The matrix representing $\hat{J}_x$ is $\frac{1}{2}\hbar\,\sigma_x$ so we need to find the eigenvalues and eigenvectors of the 2×2 matrix σ_x. Let us write

$$\sigma_x\,\chi = \rho\,\chi \qquad \text{with} \qquad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} ,$$

where ρ denotes an eigenvalue and χ the corresponding eigenvector. We find the eigenvalues by rewriting this as

$$(\sigma_x - \rho\,1)\;\chi = 0 \, ,$$

where 1 denotes the unit 2×2 matrix. This is a pair of simultaneous equations for χ_1 and χ_2, which only have a non-trivial solution if the determinant of the 2×2 coefficient matrix on the left-hand side is singular. The condition for this is

$$\det(\sigma_x - \rho\,1) = \begin{vmatrix} -\rho & 1 \\ 1 & -\rho \end{vmatrix} = 0\,,$$

which yields

$$\rho^2 - 1 = 0 \quad \text{implying that} \quad \rho = \pm 1\,.$$

The eigenvalues of $\hat{J}_x$ are thus $\pm\frac{1}{2}\hbar$ as anticipated.

More generally, the eigenvalues of $\hat{J}_x$ are written $m_x\hbar$. In this case, where $j = \frac{1}{2}$, we have $m_x = \pm\frac{1}{2}$.

Let us now find the eigenvectors corresponding to the two eigenvalues.

The case $\rho = 1$ The equation for the eigenvectors becomes:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \chi_2 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad \Rightarrow \quad \chi_2 = \chi_1\,.$$

We can pick any 2-component column matrix which satisfies this condition. In particular, a suitably normalised eigenvector which represents the state with $j = \frac{1}{2}$ and $m_x = \frac{1}{2}$ is

$$|j = \tfrac{1}{2}, m_x = \tfrac{1}{2}\rangle \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The case $\rho = -1$ In close analogy with the computation above, let us write:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = - \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \chi_2 \\ \chi_1 \end{pmatrix} = - \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad \Rightarrow \quad \chi_2 = -\chi_1\,.$$

Thus a suitably normalised eigenvector which represents the state with $j = \frac{1}{2}$ and $m_x = -\frac{1}{2}$ is

$$|j = \tfrac{1}{2}, m_x = -\tfrac{1}{2}\rangle \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Comments Let us briefly comment on the solutions found above.

- The eigenvectors corresponding to $m_x = \frac{1}{2}$ and $m_x = -\frac{1}{2}$ are orthogonal, as they must be:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0\,.$$

- The eigenvectors of $\hat{J}_x$ are expressible as linear combinations of the eigenvectors of $\hat{J}_z$:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

  and

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

  Thus if a system with $j = \frac{1}{2}$ is in an eigenstate of $\hat{J}_x$, for example with $m_x = \frac{1}{2}$, then the probability that a measurement of $\hat{J}_z$ yields the result $m = \frac{1}{2}$ is $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$.

What has emerged from this analysis is that we can consider systems with $j = 1/2$ as having *intrinsic* angular momentum, which has nothing to do with the orbital motion of the particle about some point. It is a property of the system in their own rest frame, and can be seen as an internal degree of freedom of the particle. Hence the wave function describing the state of the system must also depend on an index m labelling the values of the internal degrees of freedom. For the case $j = 1/2$, m can take two values, and therefore the wave functions have two components as discussed above. For the general case of spin j, the wave functions have $2j + 1$ components. This intrinsic angular momentum is known as *spin* and doesn't really have any classical analogue. Electrons, protons, neutrons and many of the more unstable particles have spin $\frac{1}{2}$.

The theory that we have developed for $j = \frac{1}{2}$ provides the framework for analysing the quantum mechanics of spin $\frac{1}{2}$ particles. Conventionally we write $s = \frac{1}{2}$ rather than $j = \frac{1}{2}$ when discussing such particles. The spin angular momentum operator is written $\underline{\hat{S}}$. $\hat{S}_z$ has eigenvalues $m_s\hbar$ with $m_s = \pm\frac{1}{2}$. Often these two states, with $m_s = \pm\frac{1}{2}$, are referred to as 'spin up' and 'spin down' respectively.

Of course, the wavefunction of a spin-$\frac{1}{2}$ particle also has a spatial dependence so the complete specification of the state is of the form

$$\psi = \psi_1(\underline{r})\,\alpha + \psi_2(\underline{r})\,\beta = \begin{pmatrix} \psi_1(\underline{r}) \\ \psi_2(\underline{r}) \end{pmatrix},$$

where

$$\alpha \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \beta \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The probability interpretation is then a straightforward generalisation:

> $|\psi_i(\underline{r})|^2\,\mathrm{d}\tau$ is the probability of finding the particle in the volume $\mathrm{d}\tau$ at $\underline{r}$ with z-component of spin $\frac{1}{2}\hbar$ if $i = 1$ or spin $-\frac{1}{2}\hbar$ if $i = 2$.

14.6 The Stern-Gerlach Experiment

We can now understand the result of the original Stern-Gerlach experiment, which was conducted with a beam of silver atoms and found two emergent beams, corresponding to $j = \frac{1}{2}$.

Let us now consider a more elaborate experiment involving not one but several Stern-Gerlach magnets, which we use to make *successive* measurements of various components of angular momentum.

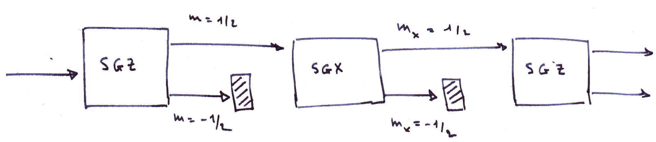

Figure 14.1: A beam of atoms,each with $j = \frac{1}{2}$, travelling along the y-axis passes through a sequence of Stern-Gerlach magnets whose mean fields are oriented along either the z-direction (SGZ) or the x-direction (SGX). The shaded boxes represent absorbers.

We assume that we can neglect any interaction between the particles in the beam. The two beams emerging from the first magnet have $m = \frac{1}{2}$ and $m = -\frac{1}{2}$, respectively, but only the former is allowed to proceed to the second magnet. Thus *we know that each particle entering the second magnet is in the state* $|j = \frac{1}{2}, m = \frac{1}{2}\rangle$, represented by the column matrix

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

The second magnet, which has its mean field aligned with the x-direction, serves to measure the x-component of angular momentum. We can predict the outcome by expanding the state $|j = \frac{1}{2}, m = \frac{1}{2}\rangle$ in eigenstates of $\hat{J}_x$ and finding the probability amplitudes for the two possible outcomes, $m_x = \frac{1}{2}$ and $m_x = -\frac{1}{2}$. Thus

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

We find the amplitudes a and b by orthogonal projection in the usual way:

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}, \\ b &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}. \end{aligned}$$

We have then for the desired probabilities

$$\begin{array}{lcl} \text{probability of getting } m_x = \tfrac{1}{2} & \text{is} & |a|^2 = \dfrac{1}{2} \\ \text{probability of getting } m_x = -\tfrac{1}{2} & \text{is} & |b|^2 = \dfrac{1}{2}. \end{array}$$

Since each particle is therefore equally likely to be found with $m_x = \frac{1}{2}$ or $m_x = -\frac{1}{2}$, equal numbers, on average, go into each of the two emergent beams and so the two beams will have *equal intensity*.

Regeneration What happens if we select only those particles with $m_x = \frac{1}{2}$ emerging from the second magnet and allow them to impinge on a third magnet whose mean field is aligned with the z-direction? This is the situation illustrated in Fig. 14.1. We are, in effect, remeasuring $\hat{J}_z$ by means of the third apparatus. We know that the state of particles entering the third magnet is $|j = \frac{1}{2}, m_x = \frac{1}{2}\rangle$ and we can expand this state in terms of the complete set of eigenstates of $\hat{J}_z$. The expansion coefficients will be the probability amplitudes required to compute the probabilities of getting the two possible outcomes $m = \frac{1}{2}$ and $m = -\frac{1}{2}$ when we measure $\hat{J}_z$ for each particle:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

and we see that the desired amplitudes are both $\frac{1}{\sqrt{2}}$, so giving equal probabilites for the two outcomes. The remarkable feature of this result is that the probability of getting $m = -\frac{1}{2}$ is non-zero *despite our having eliminated the beam with* $m = -\frac{1}{2}$ *which emerged from the first magnet!* This phenomenon is referred to as *regeneration*. It has arisen here because the second measurement, of the x-component of angular momentum, was *incompatible with the first measurement*, of the z-component.

General Remarks More generally, if the second apparatus is aligned so that its mean field lies not in the x-direction, but in the $x-z$ plane at an angle θ to the z-axis, then it measures the component of angular momentum not along the x-direction but along the direction of a unit vector

$$\underline{n} = \sin\theta\, \underline{e}_x + \cos\theta\, \underline{e}_z\,,$$

where $\underline{e}_x$ and $\underline{e}_z$ are the usual Cartesian unit vectors in the x- and z-directions respectively.

The relevant eigenstates are then those of the matrix

$$\underline{\sigma}.\underline{n} = \sigma_x \sin\theta + \sigma_z \cos\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} .$$

14.7 Summary

As usual, we summarize the main concepts introduced in this lecture.

- Half-integer values of the angular momentum. Orbital angular momentum can only have integer values of j.
- Matrix representation of the angular momentum for $j = 1/2$.
- Two-dimensional complex space of states with $j = 1/2$.
- Spin as an internal degree of freedom.
- Examples.