Lecture 15

Addition of angular momenta

15.1 Introduction

We now turn to the problem of dealing with two angular momenta. For example, we might wish to consider an electron which has both an intrinsic spin and some orbital angular momentum, as in a real hydrogen atom. Or we might have a system of two electrons and wish to know what possible values the total spin of the system can take. Classically, the angular momentum is a vector quantity, and the total angular momentum is simply $\underline{J} = \underline{J}_1 + \underline{J}_2$. The maximum and minimum values that $\underline{J}$ can take correspond to the case where either $\underline{J}_1$ and $\underline{J}_2$ are parallel, so that the magnitude of $\underline{J}$ is $|\underline{J}_1| + |\underline{J}_2|$ or antiparallel when it has magnitude $||\underline{J}_1| - |\underline{J}_2||$.

This lecture discusses the addition of angular momenta for a quantum system.

15.2 Total angular momentum operator

In the quantum case, the total angular momentum is represented by the operator

$$\underline{\hat{J}} \equiv \underline{\hat{J}}_1 + \underline{\hat{J}}_2 \,.$$

We assume that $\underline{\hat{J}}_1$ and $\underline{\hat{J}}_2$ are *independent angular momenta*, meaning each satisfies the usual angular momentum commutation relations

$$[\hat{J}_{nx}, \hat{J}_{ny}] = i\hbar\, \hat{J}_{nz}, \quad \text{etc.}, \quad [\hat{J}_n^{\;2}, \hat{J}_{ni}] \quad \text{etc.}$$

where $n = 1,\, 2$ labels the individual angular momenta, $i = x,\, y,\, z$ and etc. stands for cyclic permutations. *Furthermore, any component of* $\underline{\hat{J}}_1$ *commutes with any component of* $\underline{\hat{J}}_2$:

$$[\hat{J}_{1i}, \hat{J}_{2k}] = 0, \quad i,\, k = x,\, y,\, z\,,$$

so that the two angular momenta are *compatible*. It follows that the four operators $\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_2^2, \hat{J}_{2z}$ are *mutually commuting* and so must possess a common eigenbasis. This common eigenbasis is known as the *uncoupled basis* and is denoted $\{|j_1, m_1, j_2, m_2\rangle\}$ in Dirac notation. It has the following properties:

$$\boxed{\begin{array}{rcl} \hat{J}_1^2\, |j_1, m_1, j_2, m_2\rangle &=& j_1(j_1+1)\hbar^2\, |j_1, m_1, j_2, m_2\rangle \\ \hat{J}_{1z}\, |j_1, m_1, j_2, m_2\rangle &=& m_1\hbar\, |j_1, m_1, j_2, m_2\rangle \\ \hat{J}_2^2\, |j_1, m_1, j_2, m_2\rangle &=& j_2(j_2+1)\hbar^2\, |j_1, m_1, j_2, m_2\rangle \\ \hat{J}_{2z}\, |j_1, m_1, j_2, m_2\rangle &=& m_2\hbar\, |j_1, m_1, j_2, m_2\rangle \end{array}}\,.$$

It is easy to establish that the total angular momentum operators satisfy the usual commutation relations

$$[\hat{J}_x, \hat{J}_y] = i\hbar\, \hat{J}_z, \quad \text{etc.}, \quad [\hat{J}^2, \hat{J}_i] = 0\,.$$

As for any angular momentum operator then, $\hat{J}^2$ has eigenvalues $j(j+1)\hbar^2$ whilst the operator corresponding to the z-component of the total angular momentum has eigenvalues $m\hbar$ with m running between j and $-j$ in integer steps for a given j.

15.3 Addition Theorem

The question which then arises is, given two angular momenta, corresponding to angular momentum quantum numbers j_1 and j_2 respectively, what are the allowed values of the total angular momentum quantum number, j ? The answer is provided by the *Angular Momentum Addition Theorem*:

> The allowed values of the total angular momentum quantum number j, given two angular momenta corresponding to quantum numbers j_1 and j_2 are:
>
> $$j = j_1 + j_2,\ j_1 + j_2 - 1,\ \ldots,\ |j_1 - j_2|$$
>
> and for each of these values of j, m takes on the $(2j+1)$ values
>
> $$m = j,\ j-1, \ldots, -j\ .$$

The proof of this theorem is beyond the scope of these lectures, and is deferred to more advanced courses.

It is easy to show that $\hat{J}^2$ commutes with $\hat{J}_1^2$ and $\hat{J}_2^2$ but *not* with $\hat{J}_{1z}$ or with $\hat{J}_{2z}$ by writing

$$\hat{J}^2 \equiv (\underline{\hat{J}}_1 + \underline{\hat{J}}_2)^2 = \{\hat{J}_1^2 + \hat{J}_2^2 + 2\underline{\hat{J}}_1 \cdot \underline{\hat{J}}_2\}\ .$$

The dot product contains the x and y components of the two angular momenta which do not commute with the respective z components.

The operator $\hat{J}_z$ commutes with $\hat{J}_1^2$ and $\hat{J}_2^2$ and so the set of four operators $\hat{J}^2, \hat{J}_z, \hat{J}_1^2, \hat{J}_2^2$ are *also* a mutually commuting set of operators with a common eigenbasis known as the *coupled basis*, denoted $\{|j, m, j_1, j_2\rangle\}$ and satisfying

$$\boxed{\begin{array}{rcl} \hat{J}^2\,|j,m,j_1,j_2\rangle &=& j(j+1)\hbar^2\,|j,m,j_1,j_2\rangle \\ \hat{J}_z\,|j,m,j_1,j_2\rangle &=& m\hbar\,|j,m,j_1,j_2\rangle \\ \hat{J}_1^2\,|j,m,j_1,j_2\rangle &=& j_1(j_1+1)\hbar^2\,|j,m,j_1,j_2\rangle \\ \hat{J}_2^2\,|j,m,j_1,j_2\rangle &=& j_2(j_2+1)\hbar^2\,|j,m,j_1,j_2\rangle \end{array}}\ .$$

These are states of *definite total angular momentum and definite z component of total angular momentum* but *not* in general states with definite J_{1z} or J_{2z}. In fact, they are expressible as linear combinations of the states of the uncoupled basis, with coefficients known as *Clebsch-Gordan coefficients*, which you can find tabulated in many textbooks.

Example We consider the important case of two spin-$\frac{1}{2}$ particles for which the spin quantum numbers are $s_1 = \frac{1}{2}$ and $s_2 = \frac{1}{2}$ respectively. According to the theorem, the total spin quantum number s takes on the values $s_1 + s_2 \equiv 1$ and $|s_1 - s_2| \equiv 0$ only.

Thus two electrons can have a total spin of 1 or 0 only: these states of definite total spin are referred to as *triplet* and *singlet* states respectively, because in the former, there are three possible values of the spin magnetic quantum number, $m_s = 1,\, 0,\, -1$, whereas in the latter there is only one such value, $m_s = 0$.

The states of the uncoupled basis are

$$\alpha_1\,\alpha_2, \quad \alpha_1\,\beta_2, \quad \beta_1\,\alpha_2, \quad \beta_1\,\beta_2\,,$$

where the subscripts 1 and 2 refer to electrons 1 and 2 respectively. The operators $\hat{S}_1^2$ and $\hat{S}_{1z}$ act only on the parts labelled 1, whilst $\hat{S}_2^2$ and $\hat{S}_{2z}$ act only on the parts labelled 2.

It should be clear that that since $\alpha_1\,\alpha_2$ has $m_{s_1} = \frac{1}{2}$ and $m_{s_2} = \frac{1}{2}$ it must have $m_s = 1$, that is, total z-component of spin $\hbar$, and *can therefore only be* $s = 1$ *and not* $s = 0$. This is an example of what is known as a *stretched state*: it has the maximum possible value of the z component of total angular momentum (spin) and must therefore be a member of both the coupled and uncoupled basis: $\alpha_1\,\alpha_2 \equiv |s = 1, m_s = 1, s_1 = \frac{1}{2}, s_2 = \frac{1}{2}\rangle$.

A similar argument shows that $\beta_1\,\beta_2$ has $m_s = -1$ and thus also can only be $s = 1$, so that $\beta_1\,\beta_2 \equiv |s = 1, m_s = -1, s_1 = \frac{1}{2}, s_2 = \frac{1}{2}\rangle$. The remaining two states of the coupled basis are, however, non-trivial linear combinations of the two remaining states of the uncoupled basis:

$$\begin{aligned} |s = 1, m_s = 0, s_1 = \tfrac{1}{2}, s_2 = \tfrac{1}{2}\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\,\beta_2 + \beta_1\,\alpha_2] \\ |s = 0, m_s = 0, s_1 = \tfrac{1}{2}, s_2 = \tfrac{1}{2}\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\,\beta_2 - \beta_1\,\alpha_2]\,. \end{aligned}$$

Proof:

We apply the lowering operator, $\hat{S}_-$ for the z component of total spin to the stretched state:

$$\hat{S}_-\,\alpha_1\,\alpha_2 \equiv (\hat{S}_{1-} + \hat{S}_{2-})\,\alpha_1\,\alpha_2\,.$$

The left-hand side we write as

$$\begin{aligned} \hat{S}_-\,|s = 1, m_s = 1, s_1 = \tfrac{1}{2}, s_2 = \tfrac{1}{2}\rangle &= \sqrt{1(1+1) - 1(1-1)}\,\hbar\,|s = 1, m_s = 0, s_1 = \tfrac{1}{2}, s_2 = \tfrac{1}{2}\rangle \\ &= \sqrt{2}\,\hbar\,|s = 1, m_s = 0, s_1 = \tfrac{1}{2}, s_2 = \tfrac{1}{2}\rangle\,, \end{aligned}$$

using the usual properties of the lowering operator, whilst we write the right-hand side as

$$(\hat{S}_{1-} + \hat{S}_{2-})\,|s_1 = \tfrac{1}{2}, m_{s_1} = \tfrac{1}{2}, s_2 = \tfrac{1}{2}, m_{s_2} = \tfrac{1}{2}\rangle\,,$$

and note that

$$\begin{aligned} &\hat{S}_{1-}|s_1 = \tfrac{1}{2}, m_{s_1} = \tfrac{1}{2}, s_2 = \tfrac{1}{2}, m_{s_2} = \tfrac{1}{2}\rangle = \\ &\quad \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\,\hbar\,|s_1 = \tfrac{1}{2}, m_{s_1} = -\tfrac{1}{2}, s_2 = \tfrac{1}{2}, m_{s_2} = \tfrac{1}{2}\rangle \\ &\quad = \hbar\,|s_1 = \tfrac{1}{2}, m_{s_1} = -\tfrac{1}{2}, s_2 = \tfrac{1}{2}, m_{s_2} = \tfrac{1}{2}\rangle \equiv \hbar\,\beta_1\,\alpha_2\,, \end{aligned}$$

whilst

$$\begin{aligned}
&\hat{S}_{2-}|s_1=\tfrac{1}{2},m_{s_1}=\tfrac{1}{2},s_2=\tfrac{1}{2},m_{s_2}=\tfrac{1}{2}\rangle = \\
&\quad\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,\hbar\,|s_1=\tfrac{1}{2},m_{s_1}=\tfrac{1}{2},s_2=\tfrac{1}{2},m_{s_2}=-\tfrac{1}{2}\rangle \\
&\quad=\hbar\,|s_1=\tfrac{1}{2},m_{s_1}=\tfrac{1}{2},s_2=\tfrac{1}{2},m_{s_2}=-\tfrac{1}{2}\rangle \equiv \hbar\,\alpha_1\,\beta_2\,.
\end{aligned}$$

Equating the two sides then yields the stated result:

$$|s=1,m_s=0,s_1=\tfrac{1}{2},s_2=\tfrac{1}{2}\rangle \;=\; \frac{1}{\sqrt{2}}[\alpha_1\,\beta_2\,+\,\beta_1\,\alpha_2]\,.$$

The remaining member of the coupled basis must be a linear combination of $\alpha_1\,\beta_2$ and $\beta_1\,\alpha_2$ orthogonal to this, which we can take to be

$$|s=0,m_s=0,s_1=\tfrac{1}{2},s_2=\tfrac{1}{2}\rangle \;=\; \frac{1}{\sqrt{2}}[\alpha_1\,\beta_2\,-\,\beta_1\,\alpha_2]\,.$$

15.4 Summary

As usual, we summarize the main concepts introduced in this lecture.

- Angular momentum operator for a system of two particles.
- Choice of sets of commuting observables.
- Addition theorem.
- Examples