Lecture 2

Observables

2.1 Observing observables

We have seen at the end of the previous lecture that each dynamical variable is associated to a linear operator $\hat{O}$, and its expectation value in a quantum state can be computed:

$$\langle \Psi(t)|\hat{O}|\Psi(t)\rangle = \int dx\, \Psi(x,t)^* \hat{O} \Psi(x,t)\,, \tag{2.1}$$

when there is no ambiguity about the state in which the average is computed, we shall simply write:

$$\langle O\rangle \equiv \langle \Psi(t)|\hat{O}|\Psi(t)\rangle\,. \tag{2.2}$$

Let us now clarify how these expectation values are related to what is observed in experiments. If the observable O is measured several times under identical conditions, the results is a set of values $\{O_1, O_2, \ldots, O_n\}$.

> The expectation value defined in Eq. (2.1) is equal to the average $\sum_{k=1}^{n} O_k/n$ in the limit where the number of measurements $n \to \infty$.

The possible outcomes of experiments, O_k, are the *eigenvalues* of the operator $\hat{O}$, *i.e.* the solutions of the eigenvalue equation:

$$\boxed{\hat{O}\psi_k = O_k \psi_k\,,} \tag{2.3}$$

where ψ_k is the *eigenfunction* corresponding to the eigenvalue O_k. The eigenfunction represents the wave function of a state in which the measurement of O yields the value O_k with probability 1. To check this statement, we can compute the variance of O in the state ψ_k:

$$\begin{aligned} \mathrm{Var}_k[O] &= \langle\psi_k|\hat{O}^2|\psi_k\rangle - \langle\psi_k|\hat{O}|\psi_k\rangle^2 = \\ &= O_k^2\langle\psi_k|\psi_k\rangle - (O_k\langle\psi_k|\psi_k\rangle)^2 = 0\,, \end{aligned} \tag{2.4}$$

where we have used the fact that the eigenfunctions are normalized to one.

2.2 Hermitean operators

We stated above that every observable is represented by an operator; the correspondence is

OBSERVABLE	$\Longleftrightarrow$	OPERATOR
total energy	$\Longleftrightarrow$	$\hat{H}$
position	$\Longleftrightarrow$	$\hat{X}$
momentum	$\Longleftrightarrow$	$\hat{P}$
$\vdots$	$\Longleftrightarrow$	$\vdots$

Observables take *real* values only. Therefore we must require that the operators that represent observables have only *real* eigenvalues, since we want to identify the eigenvalues with the possible results of measurements. We can guarantee this if we only use *Hermitean* operators to represent observables.

2.2.1 Hermitean conjugate

Let us define first the *Hermitean conjugate* $\hat{O}^\dagger$ of an operator $\hat{O}$. Let $\psi(x)$ and $\phi(x)$ be arbitrary spatial wave functions (*e.g.* $\psi(x) \equiv \Psi(x,0)$ and $\phi(x) \equiv \Phi(x,0)$), then

$$\int_{-\infty}^{\infty} \phi^*(x)\, \hat{O}^\dagger\, \psi(x)\, \mathrm{d}x \equiv \left(\int_{-\infty}^{\infty} \psi^*(x)\, \hat{O}\, \phi(x)\, \mathrm{d}x \right)^* .$$

We can rewrite this relation using Dirac's notation as:

$$\boxed{\langle \phi | \hat{O}^\dagger | \psi \rangle = \Big(\langle \psi | \hat{O} | \phi \rangle \Big)^*} \tag{2.5}$$

Mathematical aside
Compare Eq. (2.5) with the more familiar expression from linear algebra:

$$O^\dagger_{ij} = O^*_{ji} \,. \tag{2.6}$$

The expressions in Eq. (2.5) are the *matrix elements* of the operator $\hat{O}$, just like O_{ij} are the matrix elements of a matrix O. The quantum states $|\phi\rangle$ and $|\psi\rangle$ are the "indices" that label the matrix elements in quantum mechanics.

Using this identification, many equations that we encounter in quantum mechanics become rather familiar.

Example Let $\hat{O} = \dfrac{d}{dx}$ then we can integrate by parts to obtain

$$\int_{-\infty}^{\infty} \psi^*(x) \frac{d}{dx}\, \phi(x)\, dx = [\phi(x)\psi^*(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi(x) \frac{d}{dx}\, \psi^*(x)\, dx$$

We can discard the constant term on the right hand side, since physically acceptable wave functions vanish at $x = \pm\infty$, and if we then take the complex conjugate of the resulting equation we obtain

$$\begin{aligned} \left(\int_{-\infty}^{\infty} \psi^*(x) \frac{d}{dx}\, \phi(x)\, dx \right)^* &= -\int_{-\infty}^{\infty} \phi^*(x) \frac{d}{dx}\, \psi(x)\, dx \\ &\equiv \int_{-\infty}^{\infty} \phi^*(x) \left(\frac{d}{dx} \right)^\dagger \psi(x)\, dx \end{aligned}$$

from the definition of Hermitean conjugate. Thus we can make the identification

$$\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}.$$

2.2.2 Hermitean operators

We can now define a *Hermitean operator*; it is an operator for which

$$\boxed{\hat{O}^\dagger \equiv \hat{O}} \tag{2.7}$$

Example Eq. (2.7) is clearly not true for all operators; $\frac{d}{dx}$ is NOT Hermitean since we have just shown that

$$\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx},$$

whereas the operator $-i\hbar\frac{d}{dx}$ IS Hermitean; the proof is straightforward and is left as an exercise.

2.3 Properties of Hermitean operators

Hermitean operators obey three properties that are very important for building the logical framework of Quantum Mechanics. We list these properties here:

1. Hermitean operators have *real* eigenvalues. The eigenvalue equation is:

   $$\hat{O}\psi_k(x) = O_k\psi_k(x) \;\; k = 1, \ldots .$$

   $\psi_k(x)$ are called eigenfunctions of $\hat{O}$, O_k are the eigenvalues. Then we have:

   $$\hat{O} = \hat{O}^\dagger \Longrightarrow O_k \in \mathbb{R}$$

2. The eigenfunctions of a Hermitean operator that belong to different eigenvalues are orthogonal.

3. If $\hat{O}$ is a Hermitean operator acting on a vector space $\mathcal{V}$, there exists an orthogonal basis of $\mathcal{V}$ made of eigenvectors of $\hat{O}$. In other words, every function $\psi(x)$ can be expanded as:

   $$\psi(x) = \sum_k c_k\psi_k(x)\,,$$

where c_k are complex coefficients, computed by taking the projection of ψ onto the states ψ_k:

$$c_k = \int dx\, \psi_k(x)^* \psi(x) = \langle \psi_k | \psi \rangle \,.$$

As you can see from the equation above, for each function $\psi(x)$ there is a set of coefficients c_k, they are the *coordinates* of the function $\psi(x)$ in the basis $\{\psi_k(x)\,,\ k = 1, \ldots\}$. Do not confuse the coefficients c_k with the eigenvalues O_k! The latter are a characteristic of the operator $\hat{O}$ and have nothing to do with the function ψ.

Mathematical aside

We shall now prove the first two properties above. The proofs are useful examples of manipulations involving operators acting on wave functions. Familiarity with these kind of manipulations is essential for solving problems in Quantum Mechanics.

1. Hermitean operators have real eigenvalues.

Proof:

Suppose $\hat{O}$ is a Hermitean operator so that $\hat{O}^\dagger = \hat{O}$, and let $\hat{O}$ have an eigenvalue O_k, with corresponding eigenfunction $\psi_k(x)$:

$$\hat{O}\, \psi_k(x) = O_k\, \psi_k(x)$$

Then

$$\int_{-\infty}^{\infty} dx\, \psi_k^*(x)\, \hat{O}\, \psi_k(x) = O_k \int_{-\infty}^{\infty} dx\, \psi_k^*(x) \psi_k(x) = O_k \int_{-\infty}^{\infty} dx\, |\psi_k(x)|^2 = O_k$$

If we take the complex conjugate of this equation, we obtain

$$\left(\int_{-\infty}^{\infty} dx\, \psi_k^*(x)\, \hat{O}\, \psi_k(x) \right)^* = O_k^*$$

but if we make use of the definition of the Hermitean conjugate, we can rewrite the left-hand side of this equation in terms of $\hat{O}^\dagger$ and use the fact that $\hat{O}^\dagger = \hat{O}$ by hypothesis:

$$\left(\int_{-\infty}^{\infty} dx\, \psi_k^*(x)\, \hat{O}\, \psi_k(x) \right)^* = \int_{-\infty}^{\infty} dx\, \psi_k^*(x)\, \hat{O}^\dagger\, \psi_k(x) = \int_{-\infty}^{\infty} dx\, \psi_k^*(x)\, \hat{O}\, \psi_k(x) \,.$$

The right-hand side is now just the integral which appears in the first equation and is equal to O_k, so we have proved that

$$O_k^* = O_k$$

thus showing that the eigenvalue O_k is real as stated. **QED.**

2. The eigenfunctions of a Hermitean operator which belong to different eigenvalues are orthogonal.

Proof:

Suppose that

$$\hat{O}\,\psi_1(x) = O_1\,\psi_1(x) \qquad \text{and} \tag{2.8}$$

$$\hat{O}\,\psi_2(x) = O_2\,\psi_2(x) \qquad \text{with } O_1 \neq O_2 \tag{2.9}$$

From Eq. (2.8) we have

$$\int_{-\infty}^{\infty} dx\,\psi_2^*(x)\,\hat{O}\,\psi_1(x) = O_1 \int_{-\infty}^{\infty} dx\,\psi_2^*(x)\,\psi_1(x) \tag{2.10}$$

whereas from Eq. (2.9)

$$\int_{-\infty}^{\infty} dx\,\psi_1^*(x)\,\hat{O}\,\psi_2(x) = O_2 \int_{-\infty}^{\infty} dx\,\psi_1^*(x)\,\psi_2(x) \tag{2.11}$$

Taking the complex conjugate of Eq. (2.11) yields on the left hand side

$$\left(\int_{-\infty}^{\infty} dx\,\psi_1^*(x)\,\hat{O}\,\psi_2(x)\right)^* \equiv \int_{-\infty}^{\infty} dx\,\psi_2^*(x)\,\hat{O}^\dagger\,\psi_1(x) = \int_{-\infty}^{\infty} dx\,\psi_2^*(x)\,\hat{O}\,\psi_1(x)\,, \tag{2.12}$$

whereas the right hand side gives

$$O_2^* \int_{-\infty}^{\infty} dx\,\psi_1(x)\,\psi_2^*(x) = O_2 \int_{-\infty}^{\infty} dx\,\psi_2^*(x)\,\psi_1(x)\,, \tag{2.13}$$

using the fact that $O_2 = O_2^*$.

Comparing with Eq. (2.10) we see that

$$O_2 \int_{-\infty}^{\infty} dx\,\psi_2^*(x)\,\psi_1(x) = O_1 \int_{-\infty}^{\infty} dx\,\psi_2^*(x)\,\psi_1(x)\,, \tag{2.14}$$

which we can rearrange to yield the result

$$(O_2 - O_1) \int_{-\infty}^{\infty} dx\,\psi_2^*(x)\,\psi_1(x) = 0\,. \tag{2.15}$$

Given that $O_2 \neq O_1$ by hypothesis, this implies that

$$\int_{-\infty}^{\infty} dx\,\psi_2^*(x)\,\psi_1(x) \equiv \tag{2.16}$$

$$\equiv \langle\psi_2|\psi_1\rangle = 0\,, \tag{2.17}$$

which is the desired result. **QED.**

A generic state ψ can be expressed as a superposition of eigenstates ψ_k:

$$\psi(x) = \sum_k c_k \psi_k(x)\,, \tag{2.18}$$

the latter can be rewritten using Dirac's notation as:

$$|\psi\rangle = \sum_k c_k |\psi_k\rangle\,. \tag{2.19}$$

The set of eigenfunctions is called a *complete set* of states, or a *basis*. You can easily prove that a generic linear combination of eigenstates is not an eigenstate.

Using the fact that eigenfunctions are orthogonal (see below), you can readily check that:

$$c_m = \langle \psi_m | \psi \rangle = \int dx\, \psi_m(x)^* \psi(x)\,, \tag{2.20}$$

and therefore given a state ψ, you can compute c_m if the eigenfunctions are known.

Given the decomposition in Eq. (2.18), the probability of finding the result O_k when measuring O in the state $\psi(x)$ is given by:

$$P_k = |c_k|^2\,. \tag{2.21}$$

Clearly the sum of probabilities should be properly normalized and therefore:

$$\sum_k P_k = \sum_k |c_k|^2 = 1\,. \tag{2.22}$$

Collapse of the wave function Another important feature of quantum mechanics is the following:

> Immediately after a measurement that gave the result O_k, the system is in the state ψ_k. The state vector has been *projected* onto the eigenstate by the process of performing the measurement.

If we want to express the same concept using equations, we can say that immediately after a measurement yielding the value O_k:

$$\psi(x) \mapsto \psi_k(x)\,, \tag{2.23}$$

$$|\psi\rangle \mapsto |\psi_k\rangle\,. \tag{2.24}$$

This is sometimes referred to as the *collapse of the wave function*. The operator that performs the state reduction is called a *projection operator*:

$$|\psi\rangle \mapsto \mathcal{P}_k|\psi\rangle\,. \tag{2.25}$$

After a measurement that yielded the value O_k, the wave function of the system coincides with the eigenfunction ψ_k. Then, as discussed below Eq. (2.3), if we perform immediately another measurement of f we will find the *same* value O_k with probability 1.

Conversely, if the wave function does not coincide with one of the eigenfunctions, then the observable f does not have a given value in the state Ψ. We can only compute the probability for each eigenvalue to be the outcome of the experiment.

Mathematical aside
A projection operator satisfies:

$$\mathcal{P}_k^2 = \mathcal{P}_k, \quad \mathcal{P}_k\mathcal{P}_l = 0,\ l \neq k\,. \tag{2.26}$$

Verify that you recognize these properties by considering the more familiar case of projectors in three-dimensional Euclidean space.

Clearly these phenomena do not have a classical analogue. The description of a physical system in quantum mechanics is radically different from the classical one. You need to practice in order to get familiar with the quantum mechanical framework.

2.4 Commutators

The product of two operators is defined as you would expect:

$$\hat{O}_1\hat{O}_2|\psi\rangle = \hat{O}_1\left(\hat{O}_2|\psi\rangle\right)\,. \tag{2.27}$$

Note that the order in which the operators are applied to the state is important! The *commutator* of two operators is:

$$\left[\hat{O}_1, \hat{O}_2\right] = \hat{O}_1\hat{O}_2 - \hat{O}_2\hat{O}_1\,. \tag{2.28}$$

In general the commutator does NOT vanish, and defines a third operator, acting on quantum states:

$$\left[\hat{O}_1, \hat{O}_2\right]|\psi\rangle = \hat{O}_1\hat{O}_2|\psi\rangle - \hat{O}_2\hat{O}_1|\psi\rangle\,. \tag{2.29}$$

2.5 Momentum operator

The *momentum operator* is defined as a differential operator:

$$\hat{P}\psi(x) = -i\hbar\frac{d}{dx}\,. \tag{2.30}$$

This is a simple realization of de Broglie's duality hypothesis. Remember that according to the wave-particle duality to each particle with momentum p we can associate a wave with wavelength h/p. A wave with a fixed wavelength is a plane wave, described by the function:

$$\psi_p(x) = \exp[ipx/\hbar]\,. \tag{2.31}$$

When we act with the operator $\hat{P}$ defined in Eq. (2.30), we see that $\psi_p(x)$ is an eigenstate of $\hat{P}$ with eigenvalue p. So the plane wave corresponds to a state with given momentum p. This justifies the definition of $\hat{P}$ as a momentum operator.

Example We have seen previously that the action of the position operator $\hat{X}$ is:

$$\hat{X}\psi(x) = x\psi(x)\,, \tag{2.32}$$

i.e. the wave function is simply multiplied by the value of x. Consider the case $\hat{O}_1 = \hat{X}$, $\hat{O}_2 = \frac{d}{dx}$. Then:

$$\hat{O}_1\hat{O}_2\psi(x) = \hat{X}\left(\frac{d}{dx}\psi(x)\right) \tag{2.33}$$

$$= x\frac{d}{dx}\psi(x)\,, \tag{2.34}$$

while

$$\hat{O}_2\hat{O}_1\psi(x) = \hat{O}_2\left(\hat{X}\psi(x)\right) \tag{2.35}$$

$$= \hat{O}_2\left(x\psi(x)\right) \tag{2.36}$$

$$= \frac{d}{dx}\left(x\psi(x)\right) \tag{2.37}$$

$$= \psi(x) + x\frac{d}{dx}\psi(x)\,. \tag{2.38}$$

Putting the two results together, we obtain for this particular choice of $\hat{O}_1$ and $\hat{O}_2$:

$$\left[\hat{O}_1, \hat{O}_2\right]\psi(x) = -\psi(x)\,, \tag{2.39}$$

i.e.

$$\boxed{\left[\hat{O}_1, \hat{O}_2\right] = -1\,.} \tag{2.40}$$

From the example above we deduce the fundamental *canonical commutation relation*:

$$\boxed{\left[\hat{X}, \hat{P}\right] = i\hbar} \tag{2.41}$$

2.6 Compatible Observables

Suppose A and B are observables and we perform the following sequence of measurements in rapid succession on a single system:

1. measure A 2. measure B 3. remeasure A

Then *if and only if* the result of 3 is *certain* to be the same as the result of 1, we say that A and B are *compatible observables*. In general, this will not be the case: the measurement of B will "spoil" the result of 1.

In a little more detail, suppose that A and B are represented by operators $\hat{A}$ and $\hat{B}$ respectively, with

$$\begin{aligned} \hat{A}\, u_i(x) &= A_i\, u_i(x) \\ \hat{B}\, v_i(x) &= B_i\, v_i(x) \end{aligned}$$

Then measurement 1 yields some eigenvalue, A_j say, of $\hat{A}$, forcing the system into the state $u_j(x)$. Measurement 2 yields B_k say, forcing the system into the state $v_k(x)$, so that measurement 3 is made with the system in the state $v_k(x)$. The only way that 3 is *certain* to yield the result A_j as obtained in 1 is if $v_k(x) \equiv u_j(x)$. For this to be true in all circumstances it must be the case that each eigenfunction $v_k(x)$ of $\hat{B}$ is identical with some eigenfunction $u_j(x)$ of $\hat{A}$. If there is no degeneracy this implies a 1-1 correspondence between the eigenfunctions of $\hat{A}$ and the eigenfunctions of $\hat{B}$. We say that $\hat{A}$ and $\hat{B}$ have *a common eigenbasis*. These properties are summarized in the so-called *compatibility theorem*.

2.6.1 The Compatibility Theorem

Given two observables, A and B, represented by Hermitean operators $\hat{A}$ and $\hat{B}$, then any one of the following three statements implies the other two:

1. A and B are compatible obervables;
2. $\hat{A}$ and $\hat{B}$ have a common eigenbasis;
3. the operators $\hat{A}$ and $\hat{B}$ commute: $[\hat{A}, \hat{B}] = 0$

Example proof:
Let us show, for instance, that $3 \Rightarrow 2$. We have

$$\begin{aligned} \hat{A}\, u_i(x) &= A_i\, u_i(x) \\ \hat{B}\, v_i(x) &= B_i\, v_i(x) \end{aligned}$$

so that for *any* eigenfunction of $\hat{A}$

$$\begin{aligned} \hat{A}\hat{B}\, u_i(x) &= \hat{B}\hat{A}\, u_i(x) \quad \text{by virtue of 3} \\ &= \hat{B}\, A_i\, u_i(x) \\ &= A_i\, \hat{B}\, u_i(x) \end{aligned}$$

Thus $\hat{B}\, u_i(x)$ is an eigenfunction of $\hat{A}$ belonging to the eigenvalue A_i. If we assume that the eigenvalues are non-degenerate, then $\hat{B}\, u_i(x)$ must be some multiple of $u_i(x)$:

$$\hat{B}\, u_i(x) = \rho\, u_i(x) \quad \text{say}$$

This just says that $u_i(x)$ is an eigenfunction of $\hat{B}$ belonging to the eigenvalue ρ, and we must have that, for some j,

$$\rho = B_j \quad \text{and } u_i(x) = v_j(x)$$

Thus any eigenfunction of the set $\{u_i(x)\}$ coincides with some member of the set $\{v_j(x)\}$. The correspondence has to be 1-1 because both sets are orthonormal; if we assume that two functions in one set coincide with a single function in the other set, we are led to a contradiction that two orthogonal functions are identical to the same function. By simply relabelling all the functions in one set we can always ensure that

$$u_1(x) = v_1(x),\ u_2(x) = v_2(x),\ u_3(x) = v_3(x), \ldots \text{etc.}$$

and this is the common eigenbasis. A more general proof, in the case where the eigenvalues are degenerate is left as an exercise in problem sheet 1.

From the theoretical point of view, we can consider two commuting observables as a *single* observable, whose measurement yields two numbers, the value of A and the value of B.

2.7 Complete sets of commuting observables

Consider an observable A, and a basis made of eigenstates of $\hat{A}$, $\{|u_1\rangle, |u_2\rangle, \ldots\}$. If all the eigenvalues are non-degenerate, each eigenvalue identifies uniquely *one* eigenstate. Hence we can label the eigenstates by their eigenvalue; if

$$\hat{A}|u_n\rangle = a_n|u_n\rangle\,, \tag{2.42}$$

then we can rename:

$$|u_n\rangle \equiv |a_n\rangle \,. \tag{2.43}$$

In this case, the observable A constitutes by itself a *complete set of commuting observables* (CSCO), i.e. the eigenvalues of $\hat{A}$ are sufficient to identify the eigenfunctions that form a basis of the space of physical states.

However this is no longer true if some of the eigenvalues are degenerate, since in this case there are several eigenfunctions corresponding to the same eigenvalues. In order to distinguish these eigenfunctions, we can introduce a second observable B, which commutes with A. According to the compatibility theorem, we can find a basis of common eigenfunctions of $\hat{A}$ and $\hat{B}$. If each pair of eigenvalues $\{a_n, b_p\}$ identifies uniquely *one* vector of the basis, then the set $\{A, B\}$ is a CSCO. If this is not the case, then there must be at least one pair $\{a_n, b_p\}$ for which there exists more than one eigenvector with these eigenvalues, i.e. there exist at least two vectors $|w_1\rangle$ and $|w_2\rangle$, such that:

$$\hat{A}|w_1\rangle = a_n|w_1\rangle, \quad \hat{B}|w_1\rangle = b_p|w_1\rangle \,, \tag{2.44}$$

$$\hat{A}|w_2\rangle = a_n|w_2\rangle, \quad \hat{B}|w_2\rangle = b_p|w_2\rangle \,. \tag{2.45}$$

$$\tag{2.46}$$

In this case specifying the values of a_n and b_p is not sufficient to identify uniquely one eigenvector, since any linear combination of $|w_1\rangle$ and $|w_2\rangle$ is also a simultaneous eigenvector of $\hat{A}$ and $\hat{B}$ with the same eigenvalues.

In this case, we add to our set of observables one more quantity C, which commutes with both A and B, and we choose a basis made of simultaneous eigenvalues of the three operators $\hat{A}, \hat{B}, \hat{C}$. If this basis is unique, then $\{A, B, C\}$ is a CSCO, and each eigenfunction in the basis is identified by the set of eigenvalues $\{a_n, b_p, c_q\}$. If not, we need to add one more observable to our set, and so on.

Let us now introduce a formal definition:

A set of observables $A, B, C, \ldots$ is called a CSCO if:

(i) all the observables commute by pairs;

(ii) specifying the eigenvalues of all the operators in the CSCO identifies a unique common eigenvector.

Given a CSCO, we can choose a basis for the space of states made of common eigenvectors of the operators associated to the obervables. Each eigenvector is uniquely identifed by the values of the eigenvalues to which it corresponds.

$$\begin{aligned}
\hat{A}|a_n, b_p, c_q, \dots\rangle &= a_n|a_n, b_p, c_q, \dots\rangle\,, \\
\hat{B}|a_n, b_p, c_q, \dots\rangle &= b_p|a_n, b_p, c_q, \dots\rangle\,, \\
\hat{C}|a_n, b_p, c_q, \dots\rangle &= c_q|a_n, b_p, c_q, \dots\rangle\,, \\
&\dots\,.
\end{aligned} \tag{2.47}$$

Mathematical aside An eigenvalue λ_n of an operator $\hat{O}$ is called g-fold degenerate if there are exactly g linearly independent eigenvectors corresponding to the same eigenvalue:

$$\exists\ |u_n^{(k)}\rangle,\ \text{such that}\ \hat{O}|u_n^{(k)}\rangle = \lambda_n|u_n^{(k)}\rangle,\ \text{for}\ k = 1, \dots, g\,. \tag{2.48}$$

Note that any linear combination

$$\sum_{k=1}^{g} c_k|u_n^{(k)}\rangle,\ \text{with}\ c_k \in \mathbb{C}\,, \tag{2.49}$$

is also an eigenstate of $\hat{O}$ with the same eigenvalue λ_n.

Given a CSCO, we can expand any generic wave function in the basis of common eigenstates labeled by the eigenvalues of the observables:

$$\begin{aligned}
|\psi\rangle &= \sum_{n,p,q} c_{n,p,q}|a_n, b_p, c_q\rangle\,, \\
\psi(x) &= \sum_{n,p,q} c_{n,p,q} u_{n,p,q}(x)\,.
\end{aligned}$$

The two expressions above are equivalent. In the first line we use Dirac's notation, while in the second we explicitly write the wave function $\psi(x)$. The functions $u_{n,p,q}(x)$ are the eigenfunctions corresponding to the kets $|a_n, b_p, c_q\rangle$, i.e. they satisfy the eigenvalue equations:

$$\begin{aligned}
\hat{A}u_{n,p,q}(x) &= a_n u_{n,p,q}(x)\,, \\
\hat{B}u_{n,p,q}(x) &= b_p u_{n,p,q}(x)\,, \\
\hat{C}u_{n,p,q}(x) &= c_q u_{n,p,q}(x)\,.
\end{aligned}$$

The modulo square of the coefficients, $|c_{n,p,q}|^2$, yields the probability of finding simultaneously the values a_n, b_p, c_q if we measure A, B, C in the state $|\psi\rangle$.

2.8 Summary

Let us conclude this chapter by summarizing the main concepts introduced in this lecture.

- The result of a measurement of an observable O is one of the eigenvalues of the linear operator $\hat{O}$.
- Properties of Hermitean operators.
- A measurement that yields a result f_k, collapses the wave function into the eigenfunction ψ_k.
- Commutators and compatible obervables.
- Complete Set of Commuting Observables.
- Degeneracy.