Quantum Mechanics

Problem Sheet 6

Basics

1. polar coordinates in two dimensions.
2. more on the harmonic oscillator.
3. a few details on the Stern-Gerlach experiment.
4. more on S-G.
5. useful problem to get some practice with the eigenstates of the H atom.
6. more practice with the H atom.

Further problems

1. H atom as a harmonic oscillator.
2. Lenz vector - you already saw this in dynamics! Compare with the classical case.
Basics

1. Consider a two-dimensional system. The position of the particle can be described equivalently by the Cartesian coordinates \((x, y)\), or the polar ones \((r, \theta)\).

Sketch a two-dimensional plane, and identify the Cartesian and polar coordinates of a point \(P\).

Find the relation between the Cartesian and the polar coordinates.

The gradient operator in two-dimensional polar coordinates is:

\[
\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta},
\]

use this expression to compute the Laplacian in polar coordinates.

In two dimensions we can define the angular momentum \(\hat{L} = \hat{X} \hat{P}_y - \hat{Y} \hat{P}_x\). Note that the angular momentum has only one component in this case!! Compute \(\hat{L}\) in polar coordinates.

Show that:

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{\hat{L}^2}{\hbar^2 r^2}.
\]

2. Write down the time-independent Schrödinger equation for the two-dimensional harmonic oscillator of frequency \(\omega\). Find the eigenvalues, eigenfunctions, and their degeneracy by separating the Cartesian coordinates \(x, y\).

Use the result in B1 to write the equation for the radial wave function of a stationary state of the two-dimensional harmonic oscillator.

\(\text{(Hint: there are several steps to be performed to get to the radial equation. Don’t give up!)}\)

3. In an experiment of the Stern-Gerlach type, a beam of atoms, each with \(\ell = 1\) and kinetic energy \(3 \times 10^{-20}\) \(J\), travels for 0.03 \(m\) through a magnetic field whose gradient \(\partial B/\partial z = 2.3 \times 10^3\) \(Tm^{-1}\). Given that the magnitude of the deflecting force on each atom is

\[F_z = \mu_z \frac{\partial B}{\partial z}\]

and that \(\mu_z = \mu_B L_z / \hbar\)

where \(\mu_B\), the so-called Bohr magneton, is \(9.27 \times 10^{-24}\) \(JT^{-1}\), calculate the transverse deflections produced at a screen 0.25 \(m\) from the exit from the magnet.

4. Can the Stern-Gerlach experiment be done with free electrons? What effect does the proton have in a hydrogen Stern-Gerlach experiment? Give quantitative answers.

5. Consider a hydrogen atom whose wavefunction at time \(t = 0\) is the following superposition of energy eigenfunctions:

\[\Psi(\mathbf{r}, t) \equiv \psi(\mathbf{r}) = \frac{1}{\sqrt{78}} [2u_{100}(\mathbf{r}) - 7u_{200}(\mathbf{r}) + 5u_{322}(\mathbf{r})]\]

- Is this wavefunction an eigenfunction of the parity operator?
- What is the probability of obtaining the result (i) \(E_1\) (ii) \(E_2\) (iii) \(E_3\), on measuring the total energy?
- What are the expectation values of the energy, of the square of the angular momentum and of the \(z\) component of the angular momentum?
6. The normalised energy eigenfunction of the ground state of the hydrogen atom \((Z = 1)\) is
\[
u_{100}(r) = C \exp(-r/a_0)
\]
where \(a_0\) is the Bohr radius and \(C\) is a normalisation constant. For this state

- calculate the normalisation constant, \(C\); you may wish to note the useful integral
\[
\int_0^\infty \exp(-br) r^n \, dr = \frac{n!}{b^{n+1}}, \quad n > -1
\]
- determine the radial distribution function, \(D_{10}(r)\), and sketch its behaviour; hence determine the most probable value of the radial coordinate, \(r\), and the probability that the electron is within a sphere of radius \(a_0\); recall that \(Y_0^0(\theta, \phi) = 1/\sqrt{4\pi}\);
- calculate the expectation value of \(r\);
- calculate the expectation value of the potential energy, \(V(r)\);
- calculate the uncertainty, \(\Delta r\), in \(r\).
Further problems

1. In the TISE for the H atom, set:

\[ r = \frac{\lambda z^2}{2}, \]
\[ \frac{\chi(r)}{r} = \frac{F(z)}{z}; \]

Show that \( F(z) \) obeys the radial equation of the two-dimensional harmonic oscillator discussed in B2.
Deduce the wave function for the ground state of the H atom.
*(This solution of the H atom is due to Schwinger).*

2. In three dimensions, consider the Lenz vector:

\[ \hat{Q} = \frac{1}{2m} \left( \hat{P} \times \hat{L} - \hat{L} \times \hat{P} \right) - \frac{e^2}{r} \hat{X}, \]

where \( X \) and \( P \) are the three-dimensional position and momentum, \( L \) is the angular momentum operator, and \( m \) the mass of a particle. Show that \( Q \) commutes with the Hamiltonian for the H atom.
Show that

\[ \hat{Q}^2 = e^2 + \frac{2\hat{H}(\hat{L}^2 + \hbar^2)}{m}. \]