Chapter 1

States of a system

1.1 Classical systems

Physical objects that obey the laws of classical mechanics are more familiar to us since we can draw examples from the direct observation of macroscopic phenomena in our daily life. For instance the trajectory of a tennis ball, or the motion of a cyclist at the Olympics, are described by the well-known Newton's law:

$$\mathbf{f} = m\,\mathbf{a}\,. \tag{1.1}$$

This remarkable equation marks the birth of modern physics: it relates the state of a physical system (kinematics) to the total force acting on the system (dynamics). Solving the differential equation yields the state of the system at all times once the initial state is known, *i.e.* the theory is *predictive* and its predictions can be tested by subsequent observations of the state of the system.

Newton's law embodies all the fundamental concepts that are used for the description of any physical system: the state of the system is described by a mathematical object, and the dynamics is encoded in a differential equation, usually called an *equation of motion*. According to the complexity of the problem under study, more or less sophisticated mathematical tools are required both to model the system and to solve the equations of motion.

We shall now discuss in more detail the quantities that appear in Newton's law. Note that this is a vectorial equality, *i.e.* vectors appear both on the LHS (the force $\mathbf{f}$) and RHS (the acceleration $\mathbf{a}$) of the equation:

- $\mathbf{a}$ is the acceleration, *i.e.* the second derivative of the position $\mathbf{x}$ of the system with respect to time, $\mathbf{a} = \frac{d^2}{dt^2}\mathbf{x}$;
- $\mathbf{f}$ is the force acting on the system;
- m is the mass of the system, this is a real number multiplying the vector $\mathbf{a}$.

1.1.1 Trajectory of a classical point particle

The state of a point particle is specified by its position as a function of time $\mathbf{x}(t)$. $\mathbf{x}(t)$ is a *vector* and spans a trajectory in d–dimensional space. A particle moving on a line, or on a plane corresponds to $d = 1, 2$ respectively, while the space we experience normally is three–dimensional. Some explicit examples can be found in the Problem section. The trajectory of a particle in a three-dimensional space is represented in Fig. 1.1. At each time t, the position of the particle is described by a three-dimensional vector $\mathbf{x}(t)$. In a given reference frame the vector $\mathbf{x}(t)$ is represented by three real coordinates $(x_1(t), x_2(t), x_3(t))$.

Let us now introduce some mathematical concepts that are useful in modelling the kinematics of a classical point particle. From a mathematical point of view, the trajectory of the particle can be seen as a *mapping* which associates to each time t a vector $\mathbf{x}$. In three dimensions the vector $\mathbf{x}$ is represented by its coordinates in a given reference frame, *i.e.* by a triplet (x_1, x_2, x_3) of real numbers; we shall introduce the following notation to denote such a mapping:

$$\begin{aligned} \mathbf{x} : \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto (x_1, x_2, x_3)\,. \end{aligned} \tag{1.2}$$

If we are only interested in the evolution of a given physical system confined in a finite box $\mathcal{B} \subset \mathbb{R}^3$, and over a finite time range $\mathcal{I} = [t_0, t_1] \subset \mathbb{R}$, then the mapping would be:

$$\mathbf{x} : \mathcal{I} \to \mathcal{B}\,. \tag{1.3}$$

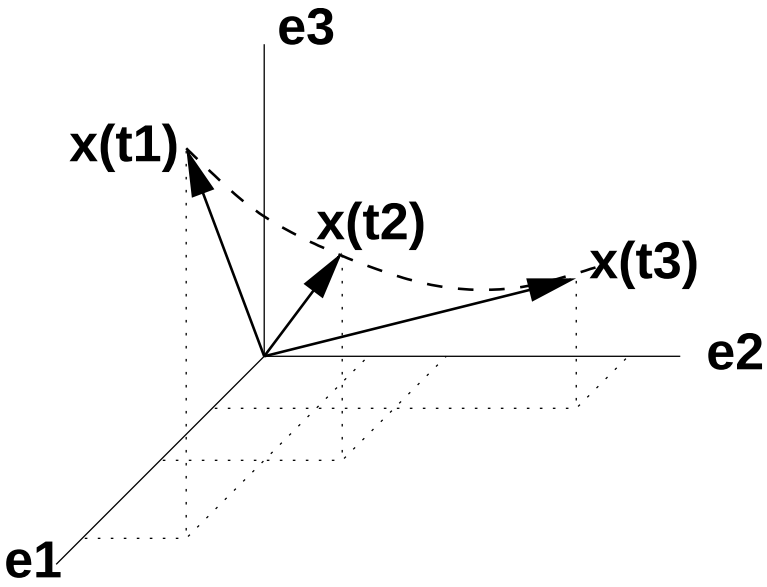

Figure 1.1: Trajectory of a point particle in three–dimensional space. The vector $\mathbf{x}(t)$ describes the position of the particle as a function of time. Three subsequent positions of the particle $\mathbf{x}(t_1), \mathbf{x}(t_2), \mathbf{x}(t_3)$ are shown in this picture. The dashed line is the trajectory of the particle. Choosing a reference frame, the vector can be described by the three coordinates $\mathbf{x} = (x_1, x_2, x_3)$. The *same* vector $\mathbf{x}$ has different coordinates if the reference frame is changed.

In general we shall indicate by t the time variable, and by $x_i(t)$ the i-th coordinate of the position vector. When the time–dependence is not explicitly needed we may simply write x_i. Derivation with respect to time is sometimes denotes by a dot:

$$\frac{d}{dt}x_i(t) \equiv \dot{x}_i(t)\,, \tag{1.4}$$

$$\frac{d^2}{dt^2}x_i(t) \equiv \ddot{x}_i(t)\,. \tag{1.5}$$

Note that, in order to model the system, we made a fundamental approximation: we *assumed* that the physical system can be described as a point particle, whose position in space is given by the vector $\mathbf{x}$. In this approximation a physical body is identified with its center of mass. Additional variables are needed if we want to describe internal degrees of freedom.

1.1.2 Equation of motion

The time-dependence of the particle position is obtained by solving Eq. 1.1. As already specified above this is a second–order differential equation relating vectors. For a d–dimensional system it can rewritten as a set of d differential equations between the components of the position and the force vectors. The solution of a second–order differential equation is completely fixed once we know the value of the function and of its first derivative at one time t_0. For instance, for a point–particle of

mass m which experiences a constant force $\mathbf{f}$:

$$x_i(t) = \frac{1}{2}\frac{f_i}{m}(t-t_0)^2 + \dot{x}_i(t_0)(t-t_0) + x_i(t_0)\,. \tag{1.6}$$

Given the position and the velocity of the system at a given time t_0, the whole trajectory is determined by Eq. 1.6.

Exercise 1.1.1 Make sure you understand *all* the symbols in the Eq. 1.6.

The solution of the equation of motion can become more difficult if the force in not constant, but the basic idea remains the same. The state of a classical system is completely determined by the knowledge of its position and velocity at a given time $(\mathbf{x}, \dot{\mathbf{x}})$. Equivalently we can replace the velocity of the particle by its momentum $\mathbf{p} = m\dot{\mathbf{x}}$, so that finally the system is described by a $2d$–dimensional vector $(\mathbf{x}, \mathbf{p})$. In three dimensions this is a six–dimensional vector. Note that in classical mechanics $\mathbf{x}$ and $\mathbf{p}$ can be measured simultaneously, so there is no obstruction in specifying the state of the system by its position and momentum.

Real vectors play a central role in describing the state of a classical system. As you have already seen in previous courses real vectors can be added, subtracted, multiplied by a constant. These properties define an abstract mathematical structure called a *vector space*. We shall now move on to the description of a quantum system and see how the same structure naturally emerges when considering the set of all possible wave functions. An abstract description of vector spaces will be presented in Sect. 1.3.

1.1.3 Coordinates

Let us consider three-dimensional space, with the usual definition of the scalar product between vectors. A triplet of vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are called orthonormal if and only if:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}\,. \tag{1.7}$$

A generic vector $\mathbf{x}$ can be uniquely written as a linear combination of the $\mathbf{e}_i$:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3\,. \tag{1.8}$$

The coordinates (x_1, x_2, x_3) are easily computed:

$$x_i = \mathbf{e}_i \cdot \mathbf{x}\,. \tag{1.9}$$

Given two vectors $\mathbf{x}, \mathbf{y}$, with respective coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3), the scalar product is given by

$$\mathbf{x} \cdot \mathbf{y} = \sum_i x_i y_i\,. \tag{1.10}$$

The *same* vector has different coordinates in different reference frames. An example of a change of reference frame can be found in Sect. 1.5.

1.2 Quantum systems

The state of a quantum mechanical system is given by its wave function. We shall see in this Section that the set of all possible wave functions (*i.e.* of all possible quantum states) shares some formal similarities with the set of states of a classical particles. We follow very closely the discussion in chapter 2 of [Cla05].

1.2.1 Probability amplitude of a quantum point particle

The wave function $\psi(\mathbf{x}, t)$ of a quantum particle is interpreted as a probability amplitude; *i.e.* $|\psi(\mathbf{x}, t)|^2 \, d^3x$ yields the probability of finding the paritcle at a time t in a volume d^3x about the point $\mathbf{x}$. Conservation of probability requires:

$$\int d^3x \, |\psi(\mathbf{x}, t)|^2 = 1 \,, \tag{1.11}$$

and therefore ψ has to be a square–integrable function. The set of square–integrable functions is called L^2.

From a physical point of view the meaningful wave functions form a smaller set than L^2; discontinuities in the wave functions do not have a clear physical interpretation, and we can safely restrict ourselves to functions that have a bounded domain (the latter describe a system that is confined to a finite region of space). We shall denote $\mathcal{F}$ the set of physically meaningful wave functions.

Linear combinations of elements of $\mathcal{F}$ also represent physical states. This fact translates into the following mathematical statement. If $\psi_1, \psi_2 \in \mathcal{F}$, then:

$$\psi(\mathbf{x}) = \lambda_1 \psi_1(\mathbf{x}) + \lambda_2 \psi_2(\mathbf{x}) \in \mathcal{F} \,, \tag{1.12}$$

for any arbitrary complex numbers λ_1, λ_2.

Exercise 1.2.1 Show that $\psi(\mathbf{x})$ defined in Eq. 1.12 is a square–integrable function.

A *scalar product* between elements of $\mathcal{F}$ can be defined. It is a mapping that associates a complex number to each ordered pair of functions:

$$(\psi, \phi) = \int d^3x \, \psi(\mathbf{x})^* \phi(\mathbf{x}) \,. \tag{1.13}$$

Exercise 1.2.2 Starting from the definition of the scalar product, verify the following properties:

$$(\psi, \phi) = (\phi, \psi)^* \,, \tag{1.14}$$

$$(\psi, \lambda_1 \phi_1 + \lambda_2 \phi_2) = \lambda_1 (\psi, \phi_1) + \lambda_2 (\psi, \phi_2) \,, \tag{1.15}$$

$$(\lambda_1 \psi_1 + \lambda_2 \psi_2, \phi) = \lambda_1^* (\psi_1, \phi) + \lambda_2^* (\psi_2, \phi) \,. \tag{1.16}$$

Note that the scalar product in anti–linear with respect to the first function of the pair.

As you get more familiar with this scalar product, you should appreciate the fact that it is a natural extension of the scalar product of real vectors. Note however that we are now dealing with a *complex* vector spaces, *i.e.* the coefficients of linear combinations are complex numbers.

Exercise 1.2.3 The scalar product allows to define the L^2 norm of a state:

$$\|\psi\|^2 = (\psi, \psi) \ . \tag{1.17}$$

Check that the norm is a real, positive number and that $\|\psi\| = 0 \Leftrightarrow \psi \equiv 0$.
Prove the *Schwartz inequality*:

$$|(\psi_1, \psi_2)| \leq \sqrt{(\psi_1, \psi_1)}\sqrt{(\psi_2, \psi_2)} \tag{1.18}$$

We have highlighted a close analogy between the states of a classical and a quantum system. In both cases the set of states has the mathematical structure of a vector space, in which a scalar product can be defined. The main difference is that classical states form a real vector space, while quantum states form a complex vector space.

1.2.2 Equation of motion

In quantum mechanics the temporal evolution of the system is given by the Schrödinger equation:

$$i\frac{\partial}{\partial t}\psi(t, \mathbf{x}) = H\psi(t, \mathbf{x}) \, . \tag{1.19}$$

This is the analog of Newton's equation: the state of the system is described by the wave function ψ, and the dynamics of the system is encoded in the Hamiltonian H. The Hamiltonian is an operator acting on $\mathcal{F}$, *i.e.* by applying H to a given wave function ψ you obtain another wave function $\psi' \in \mathcal{F}$. Note however that the Schrödinger equation is a first-order differential equation. The knowledge of $\psi(t_0, \mathbf{x})$ at a given moment in time is enough to determine its subsequent evolution - until measurements are performed.

The Schrödinger equation is solved formally by:

$$\psi(t, \mathbf{x}) = U(t, t_0)\psi(t_0, \mathbf{x}) \, , \tag{1.20}$$

where

$$U(t, t') = \exp\left[-iH(t - t')\right] \ , \tag{1.21}$$

and H is the Hamiltonian operator.

Exercise 1.2.4 Check that Eqs. 1.20, 1.21 define a solution of the Schrödinger equation.

1.2.3 Bases and coordinates in $\mathcal{F}$

We shall now develop the idea that a state in the complex vector space $\mathcal{F}$ can be described by a set of complex coordinates, which are the analog of the usual real coordinates of the three–dimensional vectors that are used to describe classical systems. In order to describe a vector by a set of coordinates we need to introduce the concept of a *basis*.

Let us consider a set of functions: $\{u_i(\mathbf{x})\}_{i\in\mathbb{N}} \in \mathcal{F}$, such that:

1. the set is orthonormal, *i.e.*
$$(u_i, u_j) = \delta_{ij}\,, \tag{1.22}$$
2. every function $\psi \in \mathcal{F}$ can be expanded in one and only one way in terms of the u_i, *i.e.*
$$\psi(\mathbf{x}) = \sum_i c_i u_i(\mathbf{x})\,. \tag{1.23}$$

The set of complex numbers c_i are said to represent ψ in the basis $\{u_i(\mathbf{x})\}$; they are the natural generalization of the concept of coordinates to the case of complex vector spaces.

Exercise 1.2.5 Using the orthonormality of the basis functions, prove that:

$$c_i = (u_i, \psi)\ . \tag{1.24}$$

It should be clear from the equation above that the coordinates c_i depend on the choice of the basis.

If $\phi(\mathbf{x}) = \sum_i b_i u_i(\mathbf{x})$ and $\psi(\mathbf{x}) = \sum_j c_j u_j(\mathbf{x})$, prove that:

$$(\psi, \phi) = \sum_i c_i^* b_i\,, \tag{1.25}$$

$$(\psi, \psi) = \sum_i |c_i|^2\ . \tag{1.26}$$

Note the analogy between the coefficients c_i and the usual concept of coordinates in $\mathbb{R}^3$. If an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is chosen, any vector $\mathbf{V}$ is represented by its coordinates (v_1, v_2, v_3):

$$\mathbf{V} = \sum_{i=1,2,3} v_i \mathbf{e}_i\,, \tag{1.27}$$

and we have:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{1.28}$$

$$v_i = \mathbf{V} \cdot \mathbf{e}_i \tag{1.29}$$

$$\mathbf{V} \cdot \mathbf{W} = \sum_i v_i w_i\,, \tag{1.30}$$

where $\cdot$ is the usual scalar product between vectors. Make sure you distinguish vectors from coordinates in the above equations. The coordinates depend on the choice of basis, while vectors are geometrical objects.

Going back to the complex vector space, we are now going to establish a relation, called the *closure relation*, which characterizes a basis $\{u_i(\mathbf{x})\}$. Let us start from the expansion Eq. 1.23, and plug it in Eq. 1.24:

$$\begin{aligned}\psi(\mathbf{x}) &= \sum_i (u_i, \psi)\, u_i(\mathbf{x}) \\ &= \sum_i \left[\int d^3\mathbf{x}' u_i(\mathbf{x}')^* \psi(\mathbf{x}') \right] u_i(\mathbf{x})\,.\end{aligned}$$

Interchanging the sum and the integral over $\mathbf{x}'$ yields:

$$\psi(\mathbf{x}) = \int d^3\mathbf{x}' \psi(\mathbf{x}') \left[\sum_i u_i(\mathbf{x}) u_i(\mathbf{x}')^* \right] . \tag{1.31}$$

Remember that Eq. 1.31 has to hold for *every* function $\psi \in \mathcal{F}$, and therefore:

$$\sum_i u_i(\mathbf{x}) u_i(\mathbf{x}')^* = \delta(\mathbf{x} - \mathbf{x}') \tag{1.32}$$

where $\delta(\mathbf{x})$ indicates the usual Dirac delta function. We shall see later that the closure relation has a very natural geometrical interpretation.

1.2.4 Bases not belonging to $\mathcal{F}$

It is often convenient to expand the wave functions of a physical system in terms of basis functions that do not belong to $\mathcal{F}$, *i.e.* functions that do not represent physical states. For simplicity we shall discuss the case of a one–dimensional system, the extension to the three–dimensional case is straightforward and is presented in one of the problems at the end of the chapter.

Plane waves

Let us consider the plane wave

$$v_p(x) = \exp\left[ipx\right] , \tag{1.33}$$

where p is a real number. It can be readily proven that the integral of $|v_p(x)|^2$ over the whole real axis diverges, and therefore $v_p \notin \mathcal{F}$. The number p can be considered as a *continuous index* labelling the functions in the set $\{v_p(x)\}$. It is an extension of the case presented in the previous Section, where we had an (potentially infinite) set of functions labelled by a discrete index i.

Expanding a generic function $\psi \in \mathcal{F}$ on the set $\{v_p(x)\}$ is nothing but the usual Fourier transform:

$$\psi(x) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, \tilde{\psi}(p) v_p(x) \tag{1.34}$$

$$\tilde{\psi}(p) = (v_p, \psi) = \int_{-\infty}^{+\infty} dx\, v_p(x)^* \psi(x)\,. \tag{1.35}$$

Compare these two formulae with Eqs. 1.23, 1.24. Having to deal with a continuous index, the summation over i is replaced by an integral over p. We choose to normalize the integrals so that each integration over p carries a factor of 2π in the denominator. The function $\tilde{\psi}(p)$ is the analog of the coefficients c_i; since p is a continuous index $\tilde{\psi}$ is a function of p. The two sets of complex numbers c_i and $\tilde{\psi}(p)$ are the coordinates of the *same* function $\psi(x)$ in two different bases, $\{u_i(x)\}$ and $\{v_p(x)\}$ respectively.

Exercise 1.2.6 Prove that:

$$(\psi, \phi) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, \tilde{\psi}(p)^* \tilde{\phi}(p) \tag{1.36}$$

$$(\psi, \psi) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, \left|\tilde{\psi}(p)\right|^2 . \tag{1.37}$$

Compare the result with Eqs. 1.25, 1.26.

The closure relation for the functions v_p is:

$$\int_{-\infty}^{+\infty} \frac{dp}{2\pi} v_p(x) v_p(x')^* = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \exp\left[ip(x-x')\right] = \delta(x-x')\,, \tag{1.38}$$

to be compared with Eq. 1.32.

And finally the orthonormality of the basis functions is expressed as:

$$(v_p, v_{p'}) = \int dx\, \exp\left[i(p-p')x\right] = (2\pi)\delta(p-p')\,, \tag{1.39}$$

which is the analog of Eq. 1.22. Note that when $p = p'$ the Dirac delta diverges, which signals the fact that our basis functions are not square–integrable.

The correspondence rules that allow us to translate from a dicrete basis to a continuous basis can be summarized as:

$$i \leftrightarrow p \tag{1.40}$$

$$\sum_i \leftrightarrow \int \frac{dp}{2\pi} \tag{1.41}$$

$$\delta_{ij} \leftrightarrow \delta(p-p') \tag{1.42}$$

Delta functions

Following the ideas developed above, we can introduce another set of basis functions $\{\xi_y(x)\}$ labelled by a continuous index y:

$$\xi_y(x) = \delta(x-y)\,. \tag{1.43}$$

This is actually a set of distributions (rather than proper functions), and they are clearly not square–integrable.

However any function $\psi \in \mathcal{F}$ can clearly be expanded in terms of the ξ:

$$\psi(x) = \int dy\, \psi(y)\xi_y(x) \tag{1.44}$$

$$\psi(y) = (\xi_y, \psi) = \int dx\, \xi_y(x)^* \psi(x)\,. \tag{1.45}$$

$$\tag{1.46}$$

The expansion can be compared again to Eqs. 1.23, 1.24, where the sum over the discrete index i is replaced by the integral over the continuous index y. Note that in this case the normalization is such that there are no factors of π associated with the integration over the continuous index. The value of the function at each point $\psi(y)$ can be interpreted as the coordinate of the function ψ in the basis provided by the delta functions $\{\xi_y(x)\}$.

Exercise 1.2.7 Check the orthonormality and the closure relation for the functions $\{\xi_y(x)\}$.

The correspondence rules in this case are:

$$i \leftrightarrow y \tag{1.47}$$

$$\sum_i \leftrightarrow \int dy \tag{1.48}$$

$$\delta_{ij} \leftrightarrow \delta(y - y') \tag{1.49}$$

The usefulness of such a basis will be clear in the following Section.

1.3 Vector spaces

We have seen in the previous Sections a number of analogies in the description of the state of a system, both in classical and quantum mechanics. These common properties are effectively summarized by introducing a mathematical construct that you have already encountered in your studies, the *vector space*, and by associating vectors to physical states. This is possible both in classical and quantum mechanics. In the classical case the vector associated to a state is a six–dimensional real vector. In quantum mechanics, we will need to introduce a more abstract notion of vectors. We shall review here some basic properties about vector spaces, for a detailed treatment you can refer to the books by Halmos [Hal58] or Tung [Tun85].

We first introduce Dirac's bra-ket notation for vectors, which will prove very powerful for formal manipulations. We will then use Dirac's notation to summarize the relevant properties of vector spaces.

1.3.1 Dirac notation

As shown in Sects. 1.2.3, and 1.2.4, the same wave function can be represented by different sets of coordinates, each one corresponding to a different choice of basis. Once the basis is specified, the coordinates – c_i or $\tilde{\psi}(\mathbf{p})$ – provide a complete description of the wave function. Even the value of the functions itself can be considered as its component in a particular basis, the δ function basis.

This is very close to what happens in classical mechanics, where the position of a system in space can be specified by the coordinates with respect to a system of axes. If the axes are changed, then the coordinates will change accordingly. However the geometrical concept of a vector allows us to perform a number of calculations and reasonings in a much simpler way.

We will push this analogy a step further and introduce the concept of a *state vector* characterizing a quantum state. The state vector belong to an abstract vector space $\mathcal{E}$ for which we need to define the notation and the rules of vector calculus.

Elements of $\mathcal{E}$: kets

Any element of $\mathcal{E}$ (*i.e.* any vector) is called a *ket*, and is represented by the symbol $|\,\rangle$. For instance we shall associate a ket $|\psi\rangle \in \mathcal{E}$ to each wave function $\psi(\mathbf{x}) \in \mathcal{F}$. In this section we review some basic properties of vector spaces using Dirac notation. It is intended to be a revision of material that you have already encountered, which gives you the opportunity to review the concepts while getting familiar with the new notation.

Definition 1.3.1 A set $\mathcal{E}$ is called a complex *vector space* if we can define two operations $+$ (*addition*) and $\cdot$ (*multiplication by a number*) which satisfy the following properties:

1. if $|\psi\rangle, |\phi\rangle \in \mathcal{E}$, then $|\psi\rangle + |\phi\rangle \in \mathcal{E}$;
2. if $|\psi\rangle \in \mathcal{E}$, and $\alpha \in \mathbb{C}$, then $\alpha|\psi\rangle = |\psi\rangle\alpha \in \mathcal{E}$;
3. there exists a *null vector* $|0\rangle$, such that: $\forall|\psi\rangle \in \mathcal{E}, |\psi\rangle + |0\rangle = |\psi\rangle$;
4. $\forall|\psi\rangle \in \mathcal{E}$, there exists $|-\psi\rangle \in \mathcal{E}$ such that: $|\psi\rangle + |-\psi\rangle = 0$;

5. the operation + is *commutative* and *associative*, *i.e.* :

$$\begin{aligned} |\psi\rangle + |\phi\rangle &= |\phi\rangle + |\psi\rangle \,, \\ (|\psi\rangle + |\phi\rangle) + |\eta\rangle &= |\psi\rangle + (|\phi\rangle + |\eta\rangle) \,; \end{aligned}$$

6. $\forall |\psi\rangle \in \mathcal{E}, 1 \cdot |\psi\rangle = |\psi\rangle$;

7. multiplication by a number is *associative*, *i.e.*

$$\alpha \cdot \beta \cdot |\psi\rangle = (\alpha\beta) \cdot \psi\rangle \,;$$

8. the two operations satisfy the *distributive* properties:

$$\begin{aligned} (\alpha + \beta) \cdot |\psi\rangle &= \alpha \cdot |\psi\rangle + \beta \cdot |\psi\rangle \,, \\ \alpha \cdot (|\psi\rangle + |\phi\rangle) &= \alpha \cdot |\psi\rangle + \alpha \cdot |\phi\rangle \,. \end{aligned}$$

This definition encodes in an abstract formalism all the intuitive properties of three–dimensional vectors. The advantage of having an abstract formalism is that it can be used to analyze situations where the intuitive picture is less clear.

Examples of vector spaces are: (i) the space $\mathbb{C}$ of all complex numbers; (ii) the space of all vectors in d–dimensional Euclidean space; (iii) all polynomials of a real variable t with complex coefficients and of order n or less; (iv) all functions of a real variable t which are solutions to a linear homogenous differential equation (such as the solutions of the equations of motion in classical and quantum mechanics). In each case, the operations + and $\cdot$ are defined in the conventional way.

For each of the examples in this list you chould check that you can identify the elements of the vector space, *i.e.* **the vectors, that you can sum vectors, that you can multiply the vectors by a complex number, and that the above properties are all verified!!**

We shall now give some more definitions, and state without proof some properties of vector spaces that are going to be useful in the rest of the course. The interested reader can look for detailed proofs in the references cited above.

Definition 1.3.2 A set of vectors $\{|\phi_i\rangle \in \mathcal{E}, i = 1, \ldots, m\}$ are *linearly independent* if $\sum_i \alpha_i \cdot |\phi_i\rangle$ necessarily implies $\alpha_i = 0, \forall i$.

In a set of linearly dependent vectors, some vector of the set can be written as a linear combination of the other vectors. In the familiar case of two–dimensional Euclidean space (*i.e.* the plane of the blackboard!!), any three vectors are linearly dependent, two vectors are linearly dependent if and only if they are collinear.

The concept of a basis can be extended to an abstract vector space in a natural way.

Definition 1.3.3 A set of vectors $\{|\phi_i\rangle \in \mathcal{E}, i = 1, \ldots, m\}$ forms a *basis* of a vector space $\mathcal{E}$ if: (i) they are linearly independent; and (ii) every $|\psi\rangle \in \mathcal{E}$ can be written as a linear combination of the elements of the basis:

$$|\psi\rangle = \sum_i \psi_i |\phi_i\rangle$$

The (complex) numbers ψ_i are the *coordinates* of the vector $|\psi\rangle$ with respect to the basis $\{|\phi_i\rangle\}$. Vector spaces which have a basis with a finite number of elements are called *finite dimensional*. Note that the definition above is completely generic, *i.e.* it can be applied to any vector space, without any detailed knowledge about the precise nature of the elements of the vector space. The interested reader can try to figure out bases for the vector spaces in the list of examples above.

Theorem 1.3.1 All bases of a finite dimenionsal vector space have the same number of elements.

And therefore we can define the *dimension* of a vector space:

Definition 1.3.4 The number of elements n in a basis of a finite dimensional vector space $\mathcal{E}$ is called the *dimension* of $\mathcal{E}$.

When dealing with representation theory, we will use extensively the concept that two vector spaces have the same structure. This idea is encoded in the following definition:

Definition 1.3.5 Two vector spaces $\mathcal{E}$ and $\mathcal{E}'$ are said to be *isomorphic* to each other if there exists a one–to–one mapping:

$$\begin{array}{ccc} \mathcal{E} & \rightarrow & \mathcal{E}' \\ |\psi\rangle & \mapsto & |\psi'\rangle \end{array}$$

which respects the two operations:

$$\forall |\psi\rangle, |\phi\rangle \in \mathcal{E}, \forall \alpha \in \mathbb{C}, \quad |\psi\rangle + \alpha \cdot |\phi\rangle \mapsto |\psi'\rangle + \alpha \cdot |\phi'\rangle \tag{1.50}$$

As an example, you can show that the two–dimensional Euclidean space is isomorphic to the set of complex numbers $\mathbb{C}$. These properties are usually proven by constructing explicitly the mapping between the two vector spaces.

Finally we need to be able to consider smaller vector spaces, that are made of subsets of an original one, and bigger vector spaces that can be built out of several smaller ones. For this purpose we define two new concepts.

Definition 1.3.6 A subset $\mathcal{E}_n$ of a vector space $\mathcal{E}$ which is itself a vector space, with the same definitions of the two operations, is called a subspace of $\mathcal{E}$.

A simple example is the two–dimensional Euclidean space, which can be viewed as a subspace of a higher–dimensional Euclidean space.

Theorem 1.3.2 Given a vector space $\mathcal{E}_n$ and a subspace $\mathcal{E}_m$ of respective dimensions n and m $(m < n)$, one can always choose a basis $\{|\phi_i\rangle, i = 1, \ldots, n\}$ in such a way that the first m basis vectors lie in $\mathcal{E}_m$.

Again you can check the theorem in the simple case of three- and two–dimensional Euclidean spaces.

Definition 1.3.7 Let $\mathcal{E}_1, \mathcal{E}_2$ be subspaces of $\mathcal{E}$, we say that $\mathcal{E}$ is the *direct sum* of $\mathcal{E}_1, \mathcal{E}_2$, denoted as $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$, if (i) $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$, and (ii) every vector $|\psi\rangle \in \mathcal{E}$ can be written as $|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$, where $|\psi_1\rangle \in \mathcal{E}_1$, and $|\psi_2\rangle \in \mathcal{E}_2$.

The construction of the direct sum of two arbitrary vector spaces is presented in Sect. 1.5.

Elements of the dual space: bras

Given a vector space $\mathcal{E}$ we can define a *linear functional* χ as a linear operation that associates a complex number with every element of $\mathcal{E}$:

$$\begin{aligned} \chi : \mathcal{E} &\rightarrow \mathbb{C} \\ |\psi\rangle &\mapsto \chi(|\psi\rangle) \quad \text{(a complex number)} \\ \chi(\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle) &\mapsto \alpha_1\chi(|\psi_1\rangle) + \alpha_2\chi(|\psi_2\rangle)\,. \end{aligned}$$

Note that the linear functional associates to each a ket a complex number. This is different from a linear operator which, as we will se later, associates to each ket another ket.

The set of linear functionals defined on $\mathcal{E}$ constitutes a vector space, called the *dual space* and denoted $\mathcal{E}^*$.

The elements of the dual space are called *bra vectors*, or simply *bra*. We shall use the $\langle\,|$ notation to indicate a bra. The bra $\langle\chi|$ designates the functional χ introduced above, and we shall use the following notation for the number obtained when the functional χ acts on a ket:

$$\chi\left(|\psi\rangle\right) = \langle\chi|\psi\rangle\,. \tag{1.51}$$

If there exists a scalar product in $\mathcal{E}$ [1], we can associate to every ket $|\phi\rangle \in \mathcal{E}$ an element of $\mathcal{E}^*$ denoted $\langle\phi|$. Indeed for a given ket $|\phi\rangle$ we can use the scalar product to associate to each ket $|\psi\rangle$ a complex number given by the scalar product $(|\phi\rangle, |\psi\rangle)$. It is easy to realize that this prescription defines a linear functional on $\mathcal{E}$:

$$\langle\phi|\psi\rangle = (|\phi\rangle, |\psi\rangle)\ . \tag{1.52}$$

Note that the converse property is not true, *i.e.* it is not necessarily true that one can associate a ket in $\mathcal{E}$ for each bra in $\mathcal{E}^*$.

Exercise 1.3.1 Show that the correspondence between bra and ket is *antilinear*, *i.e.* the bra associated with the ket $\alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle$ is $\alpha_1^*\langle\phi_1| + \alpha_2^*\langle\phi_2|$.
Hint: remember that the scalar product of complex vectors satisfies:

$$(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^* \tag{1.53}$$

Note that if α is a complex number and $|\psi\rangle$ is a ket, then by definition $\alpha \cdot |\psi\rangle$ is a ket. When there is no ambiguity we shall often omit the '$\cdot$' symbol, and sometimes we will use the notation:

$$|\alpha\psi\rangle = \alpha|\psi\rangle\,. \tag{1.54}$$

One must remember when dealing with bras that $\langle\alpha\psi|$ represents the bra associated with $|\alpha\psi\rangle$. Since the correspondence is antilinear:

$$\langle\alpha\psi| = \alpha^*\langle\psi| \tag{1.55}$$

[1] A vector space $\mathcal{E}$ in which a scalar product is defined is called a *Hilbert space*.

Finally notice that the Dirac notation provides us with an alternative notation for the scalar product. The usual properties of the scalar product are summarized in Dirac notation as follows:

$$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^* \tag{1.56}$$
$$\langle\phi|\alpha_1\psi_1 + \alpha_2\psi_2\rangle = \alpha_1\langle\phi|\psi_1\rangle + \alpha_2\langle\phi|\psi_2\rangle \tag{1.57}$$
$$\langle\alpha_1\phi_1 + \alpha_2\psi_2|\psi\rangle = \alpha_1^*\langle\phi_1|\psi\rangle + \alpha_2^*\langle\phi_2|\psi\rangle\,. \tag{1.58}$$

Henceforth we shall use Dirac notation for scalar products.

1.3.2 Linear operators

Linear operators acting on vector spaces play a key role in defining the action of a symmetry transformation on a physical system. We shall focus on these later on in the course. It is therefore essential that you get familiar with these concepts now.

A *linear operator* A is a linear mapping from $\mathcal{E}$ into $\mathcal{E}$:

$$\begin{array}{rccl} A: & \mathcal{E} & \to & \mathcal{E} \\ & |\psi\rangle & \mapsto & |\psi'\rangle = A|\psi\rangle\,; \end{array}$$

the linearity of the operator means that:

$$A\left(\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle\right) = \alpha_1 A|\psi_1\rangle + \alpha_2|\psi_2\rangle\,. \tag{1.59}$$

Note that, if we just follow the rule of juxtaposition of symbols, the expression:

$$|\psi\rangle\langle\chi| \tag{1.60}$$

defines an operator. Choose an arbitrary ket $|\phi\rangle$ and consider:

$$|\psi\rangle\langle\chi|\phi\rangle\,; \tag{1.61}$$

we know that $\langle\chi|\phi\rangle$ is a complex number, and therefore the whole expression in Eq. 1.61 is a ket proportional to $|\psi\rangle$. The expression $|\psi\rangle\langle\chi|$ maps a vector in $\mathcal{E}$ into another vector in $\mathcal{E}$, and therefore is an operator.

Exercise 1.3.2 Prove that the operator $|\psi\rangle\langle\chi|$ is linear.

Let $|\psi\rangle$ be a ket normalized to one, and consider the operator:

$$P_\psi = |\psi\rangle\langle\psi|\,. \tag{1.62}$$

When acting with P_ψ on a vector $|\phi\rangle$, we obtain:

$$P_\psi|\phi\rangle = |\psi\rangle\langle\psi|\phi\rangle\,, \tag{1.63}$$

i.e. a ket proportional to $|\psi\rangle$, whose norm is given the scalar product of $|\phi\rangle$ and $|\psi\rangle$. Therefore P_ψ has a clear geometrical interpretation: it is the *projector* onto the vector $|\psi\rangle$.

Exercise 1.3.3 Check that P_ψ is indeed a projector by proving that $P_\psi^2 = P_\psi$. Note that the normalization of the ket $|\psi\rangle$ is important in this step.
Let $\{|\phi_i\rangle, i = 1, \ldots, q\}$ be a set of orthnormal vectors, prove that

$$P_q = \sum_{i=1}^{q} |\phi_i\rangle\langle\phi_i| \tag{1.64}$$

is a projector by computing P_q^2. Find a geometrical interpretation for P_q.

Given a linear operator acting on a vector space $\mathcal{E}$, we can define its action on the dual space $\mathcal{E}^*$.

$$\begin{array}{rccc} A: & \mathcal{E}^* & \to & \mathcal{E}^* \\ & \langle\phi| & \mapsto & \langle\phi|A\,, \end{array}$$

where the bra $\langle\phi|A$ is defined by:

$$(\langle\phi|A)\,|\psi\rangle = \langle\phi|\,(A|\psi\rangle)\ . \tag{1.65}$$

Note that in the RHS the operator A acts on a ket, while the LHS *defines* the action of the same operator on a bra. The operator A associates with every bra $\langle\phi|$ a new bra $\langle\phi|A$.

Exercise 1.3.4 Prove that the mapping is linear, *i.e.*

$$(\alpha_1\langle\phi_1| + \alpha_2\langle\phi_2|)\,A = \alpha_1\langle\phi_1|A + \alpha_2\langle\phi_2|A\,. \tag{1.66}$$

The correspondence between kets and bras enables us to associate to each operator A another operator $A^\dagger$, called the *adjoint* operator or the *Hermitian conjugate* operator. The adjoint operator is defined by its action on the dual space, *i.e.* by the way it tranforms any given bra into some other bra.

Let $|\psi\rangle$ be an arbitrary vector in $\mathcal{E}$, the operator A associates to it a new ket $|\psi'\rangle = A|\psi\rangle \in \mathcal{E}$. To the ket $|\psi\rangle$ corresponds the bra $\langle\psi|$, and similarly the bra $\langle\psi'|$ is associated to $|\psi'\rangle$. The operator $A^\dagger$ is defined as the operator that associates $\langle\psi'|$ to $\langle\psi|$:

$$|\psi'\rangle = A|\psi\rangle \Longleftrightarrow \langle\psi'| = \langle\psi|A^\dagger\,. \tag{1.67}$$

Exercise 1.3.5 Show that $\langle\psi|A^\dagger|\phi\rangle = \langle\phi|A|\psi\rangle^*$.

Following the notation in Eq. 1.54 we can write:

$$|A\psi\rangle = A|\psi\rangle \tag{1.68}$$

$$\langle A\psi| = \langle\psi|A^\dagger\,. \tag{1.69}$$

Note that when the operator A is taken out of a bra symbol, it must be replaced by its adjoint *placed to the right* of the bra.

Exercise 1.3.6 Prove the following properties:

$$\left(A^\dagger\right)^\dagger = A \tag{1.70}$$

$$(\alpha A)^\dagger = \alpha^* A^\dagger \tag{1.71}$$

$$(A+B)^\dagger = A^\dagger + B^\dagger \tag{1.72}$$

$$(AB)^\dagger = B^\dagger A^\dagger \tag{1.73}$$

These properties are essential for the manipulations that will appear everywhere in the rest of the course.

Finally an operator is said to be *Hermitean* if:

$$A^\dagger = A\,. \tag{1.74}$$

In order to get more familiar with adjoint operators and the way they acto on kets and bras, you should try the following Exercise.

Exercise 1.3.7 Prove the following properties:

1. if A is Hermitean, $\langle\psi|A|\phi\rangle = \langle\phi|A|\psi\rangle^*$.
2. if A is Hermitean, its eigenvalues are real. The eigenvectors corresponding to different eigenvalues are orthogonal.
3. if A is Hermitean, $\langle A\psi|\phi\rangle = \langle\psi|A\phi\rangle$.
4. the projector P_ψ is Hermitean.
5. if A, B are both Hermitean, the product AB is Hermitean only if $[A,B]=0$.

1.4 The R and P representations

We will now focus on the space of physical wave functions $\mathcal{F}$, and discuss the construction of the associated vector space in more detail. In doing so, we will introduce two important basis for the vector space of physical states, that define respectively the R and P representations of the state of a quantum system.

We shall start from the space of physical wave functions $\mathcal{F}$ and construct the vector space $\mathcal{E}$ of associated kets. We assume that we can associate a ket $|\psi\rangle$ to each wave function $\psi(\mathbf{x})$, and that the scalar product of two kets is given by the scalar roduct of the wave functions, *i.e.* we *assume*':

$$\langle\phi|\psi\rangle = \int d^3x\, \phi(\mathbf{x})^* \psi(\mathbf{x})\,. \tag{1.75}$$

1.4.1 Choice of bases

As discussed in Sect. 1.2.4, we can expand the wave functions in $\mathcal{F}$ in terms of basis functions that do not belong to $\mathcal{F}$ itself:

$$\xi_{\mathbf{y}}(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{y})\,, \tag{1.76}$$

$$v_{\mathbf{p}}(\mathbf{x}) = \exp\left[i\mathbf{p}\cdot\mathbf{x}\right]\,. \tag{1.77}$$

We shall associate kets to these functions, and denote them by $|\mathbf{y}\rangle$, and $|\mathbf{p}\rangle$ respectively.

The sets $\left\{|\mathbf{y}\rangle, \mathbf{y}\in\mathbb{R}^3\right\}$, and $\left\{|\mathbf{p}\rangle, \mathbf{p}\in\mathbb{R}^3\right\}$, are characterized respectively by the continuum indices $\mathbf{y}$, and $\mathbf{p}$. Using the definitions in Eqs. (1.76, 1.77), and the definition of the vector product in $\mathcal{E}$ as defined in Eq. (1.75), it can be readily shown that these vectors are orthonormal:

$$\langle\mathbf{y}|\mathbf{y}'\rangle = \int dx^3\, \xi_{\mathbf{y}}(\mathbf{x})^* \xi_{\mathbf{y}'}(\mathbf{x}) = \delta(\mathbf{y}-\mathbf{y}')\,; \tag{1.78}$$

and satisfy the closure relation:

$$\int d^3y\, |\mathbf{y}\rangle\langle\mathbf{y}| = 1\,. \tag{1.79}$$

Note that in the closure relation we have to sum over the whole basis, ie in this case we need to integrate over the continuous index used to indentify the basis functions.

Exercise 1.4.1 Check that you understand the orthonormality and the closure relation for the $|\mathbf{y}\rangle$ kets.

Write down and prove the analog relations for the vectors $|\mathbf{p}\rangle$.

An arbitrary ket $|\psi\rangle$ can be expanded in any of the two bases. Starting with the basis $\left\{|\mathbf{y}\rangle, \mathbf{y}\in\mathbb{R}^3\right\}$, and using the closure relation above, we can write:

$$|\psi\rangle = \int d^3y |\mathbf{y}\rangle\langle\mathbf{y}|\psi\rangle\,, \tag{1.80}$$

and the coefficients of the expansion can be computed explicitly:

$$\langle \mathbf{y}|\psi\rangle = \int d^3x\, \xi_{\mathbf{y}}(\mathbf{x})^* \psi(\mathbf{x})\,, \tag{1.81}$$

$$= \psi(\mathbf{y})\,. \tag{1.82}$$

Hence we have shown a very important fact, namely that the value of the wave function at the point $\mathbf{y}$ is simply the component of the ket $|\psi\rangle$ along the basis vector $|\mathbf{y}\rangle$. The wave function is the "coordinate" of the vector describing the state of a quantum system in a particular basis, this choice of basis is called the *R–representation*. A quantum state is described by a vector in a Hilbert space $\mathcal{E}$: the fact that the vector can be represented by a wave function (*i.e.* by a complex–valued function defined over the whole three–dimensional space) corresponds to a particular choice of basis in $\mathcal{E}$.

The analog of Eq. 1.82 for the expansion in the $|\mathbf{p}\rangle$ basis can be easily computed and is left as an Exercise. Choosing the $|\mathbf{p}\rangle$ basis yields the so–called *P–representation*.

Exercise 1.4.2 Show that:

$$\langle \mathbf{p}|\psi\rangle = \tilde{\psi}(\mathbf{p})\,, \tag{1.83}$$

where $\tilde{\psi}(\mathbf{p})$ is the Fourier transform of ψ, computed at $\mathbf{p}$, and $\mathbf{p}$ is the vector that labels the ket $|\mathbf{p}\rangle$. *(Hint: you need to remember the definition of the Fourier transform of a function ψ to solve this problem!)*

Thus the wave function in momentum space $\tilde{\psi}$ can be interpreted as the component of the *same* quantum state along a different basis vector. The same state is characterized either by the wave function in position space, or the wave function in momentum space. They are like different coordinates describing the same vector in different reference frames.

Exercise 1.4.3 Using the closure relation in the R–representation, rederive the expression for the scalar product of two vectors in $\mathcal{E}$.

Changing from one representation to another is easily accomplished using the closure formula, *viz.*

$$\langle \mathbf{x}|\psi\rangle = \int d^3p\, \langle \mathbf{x}|\mathbf{p}\rangle\langle \mathbf{p}|\psi\rangle\,; \tag{1.84}$$

the quantity $\langle \mathbf{x}|\mathbf{p}\rangle$ is a scalar product, *i.e.* a complex number, which can be readily computed from the expressions for $\xi_{\mathbf{x}}$ and $v_{\mathbf{p}}$, $\langle \mathbf{x}|\mathbf{p}\rangle = \exp[i\mathbf{p}\cdot\mathbf{x}]$.

Exercise 1.4.4 Find the expression that relates the matrix element $\langle \mathbf{x}'|A|\mathbf{x}\rangle$ of an operator A in the R–representation to the matrix element $\langle \mathbf{p}'|A|\mathbf{p}\rangle$ in the P–representation.

1.4.2 The R and P operators

The R– and P–representations allow us to define the operators R and P respectively.

Let us consider a state vector $|\psi\rangle$, represented by a wave function $\langle \mathbf{r}|\psi\rangle = \psi(\mathbf{r}) = \psi(x,y,z)$, we can define the action of the operator X as follows. When acting with X on $|\psi\rangle$ we obtain the vector $|\psi'\rangle$:

$$|\psi'\rangle = X|\psi\rangle\,. \tag{1.85}$$

The vector $|\psi'\rangle$ is represented in the R–representation by the wave function $\psi'(x,y,z)$ such that:

$$\psi'(x,y,z) = x\psi(x,y,z)\,. \tag{1.86}$$

In the R–representation, the operator X is the operator that multiplies the wave function at any point (x,y,z) by x.

Note that X is an operator acting on $\mathcal{E}$, even though we have defined it by specifying its action on the wave functions. We can introduce in a similar way the operators Y, and Z:

$$\begin{aligned}
\langle \mathbf{r}|X|\psi\rangle &= x\langle \mathbf{r}|\psi\rangle\,, && (1.87)\\
\langle \mathbf{r}|Y|\psi\rangle &= y\langle \mathbf{r}|\psi\rangle\,, && (1.88)\\
\langle \mathbf{r}|Z|\psi\rangle &= z\langle \mathbf{r}|\psi\rangle\,. && (1.89)\\
& && (1.90)
\end{aligned}$$

The numbers x,y,z are precisely the continuum indices that label the ket $|\mathbf{r}\rangle$.

The R–representation is particularly suited for manipulations of the X operator since we obtain a simple number whenever the X operator acts on vector $\mathbf{r}$. As an example consider the following matrix element:

$$\begin{aligned}
\langle \phi|X|\psi\rangle &= \int d^3x\,\langle \phi|\mathbf{x}\rangle\langle \mathbf{x}|X|\psi\rangle && (1.91)\\
&= \int d^3x\,\phi(\mathbf{x})^*\,x\,\psi(\mathbf{x})\,, && (1.92)
\end{aligned}$$

where we used the closure relation for the $|\mathbf{x}\rangle$ basis in the first line of the calculation. It is important that you practice this kind of manipulations.

In a similar way we define the operators P_x, P_y, P_z by their action on the components of $|\psi\rangle$ in the P–representation:

$$\begin{aligned}
\langle \mathbf{p}|P_x|\psi\rangle &= p_x\langle \mathbf{p}|\psi\rangle\,, && (1.93)\\
\langle \mathbf{p}|P_y|\psi\rangle &= p_y\langle \mathbf{p}|\psi\rangle\,, && (1.94)\\
\langle \mathbf{p}|P_z|\psi\rangle &= p_z\langle \mathbf{p}|\psi\rangle\,. && (1.95)\\
& && (1.96)
\end{aligned}$$

Using the closure relation for the $|\mathbf{p}\rangle$ basis, and some well–known relations about derivatives and Fourier transforms, you should be able to solve the following Exercise.

Exercise 1.4.5 Prove that

$$\langle \mathbf{r}|P_x|\psi\rangle = \frac{1}{i}\frac{\partial}{\partial x}\psi(\mathbf{r})\,. \tag{1.97}$$

This result should be familiar from the quantum mechanics course, and identifies P with the momentum operator.

1.5 Problems

1.5.1 Classical trajectories

Draw the trajectory of a point particle in two–dimensional space whose trajectory is given by the following equations:

$$x(t) = t\,, \tag{1.98}$$
$$y(t) = t^2\,; \tag{1.99}$$

$$x(t) = t^2\,, \tag{1.100}$$
$$y(t) = t\,; \tag{1.101}$$

$$x(t) = \cos(2\pi t)\,, \tag{1.102}$$
$$y(t) = \sin(2\pi t)\,. \tag{1.103}$$

1.5.2 Change of coordinates

Let us consider a point particle in two–dimensional space, and let us choose a reference frame defined by two vectors orthonormal vectors $(\mathbf{e}_1, \mathbf{e}_2)$. A generic vector $\mathbf{x}$ can be written as:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2\,. \tag{1.104}$$

Let us now choose a different basis:

$$\begin{aligned} \mathbf{e}'_1 &= a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 \\ \mathbf{e}'_2 &= a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2\,. \end{aligned} \tag{1.105}$$

Show that the vectors of the new basis are orthonormal if we choose:

$$a_{11} = \cos\theta \qquad a_{12} = \sin\theta\,, \tag{1.106}$$
$$a_{21} = -\sin\theta \qquad a_{22} = \cos\theta\,. \tag{1.107}$$

The coordinates of the vector $\mathbf{x}$ in the new reference frame are (x'_1, x'_2) such that:

$$\mathbf{x} = x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2\,, \tag{1.108}$$

find an expression for (x'_1, x'_2) as functions of (x_1, x_2).

1.5.3 Plane waves in three dimensions

The three-dimensional plane waves are given by:

$$v_{\mathbf{p}}(\mathbf{x}) = \exp\left[i\mathbf{p}\cdot\mathbf{x}\right]\,. \tag{1.109}$$

Check that they are orthonormal, *i.e.*

$$(v_{\mathbf{p}}, v_{\mathbf{p}'}) = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\,. \tag{1.110}$$

The Fourier transfom in this case is written as:

$$\psi(\mathbf{x}) = \int_{-\infty}^{+\infty} \frac{d^3p}{(2\pi)^3}\, \tilde{\psi}(\mathbf{p}) v_{\mathbf{p}}(\mathbf{x})\,, \tag{1.111}$$

$$\tilde{\psi}(\mathbf{p}) = (v_{\mathbf{p}}, \psi) = \int_{-\infty}^{+\infty} d^3x\, v_{\mathbf{p}}(\mathbf{x})^* \psi(\mathbf{x})\,. \tag{1.112}$$

Prove that:

$$(\psi, \phi) = \int_{-\infty}^{+\infty} \frac{d^3p}{(2\pi)^3}\, \tilde{\psi}(\mathbf{p})^* \tilde{\phi}(\mathbf{p})\,. \tag{1.113}$$

Verify the closure relation:

$$\int_{-\infty}^{+\infty} \frac{d^3p}{(2\pi)^3} v_{\mathbf{p}}(\mathbf{x}) v_{\mathbf{p}}(\mathbf{x}')^* = \delta(\mathbf{x} - \mathbf{x}')\,. \tag{1.114}$$

Therefore the three–dimensional plane-waves can be seen as a continuous orthonormal and complete basis. The correspondence rules that allow us to translate from a dicrete basis to a continuous basis can be summarized as:

$$i \leftrightarrow \mathbf{p} \tag{1.115}$$

$$\sum_i \leftrightarrow \int \frac{d^3p}{(2\pi)^3} \tag{1.116}$$

$$\delta_{ij} \leftrightarrow \delta(\mathbf{p} - \mathbf{p}') \tag{1.117}$$

1.5.4 Direct sum of vector spaces

Given two vector spaces V_1, and V_2, let us consider the set:

$$V = \{(|x_1\rangle, |x_2\rangle), |x_1\rangle \in V_1, |x_2\rangle \in V_2\}\ , \tag{1.118}$$

and define the two operations:

$$(|x_1\rangle, |x_2\rangle) + (|y_1\rangle, |y_2\rangle) = (|x_1\rangle + |y_1\rangle, |x_2\rangle + |y_2\rangle)\,, \tag{1.119}$$

$$\alpha(|x_1\rangle, |x_2\rangle) = (\alpha|x_1\rangle, \alpha|x_2\rangle)\,. \tag{1.120}$$

Check that V is indeed a vector space called the *direct sum* of V_1 and V_2:

$$V = V_1 \oplus V_2\,. \tag{1.121}$$

The subspaces of V consisting of $(|x_1\rangle, |0\rangle)$ and $(|0\rangle, x_2\rangle$ are respectively isomorphic to V_1 and V_2.

1.5.5 Properties of operators

- The geometric series

$$f(x) = \sum_{n=0}^{+\infty} x^n \tag{1.122}$$

is know to converge to $f(x) = 1/(1-x)$, if $|x| < 1$. Let us consider an Hermitean operator A. Using a basis of eigenvectors of A, discuss the under which conditions the equality

$$f(A) = \sum_{n=0}^{+\infty} A^n = (1-A)^{-1} \tag{1.123}$$

holds.

- Let us call $|u_n\rangle$ the eigenvectors an operator B, with eigenvalues λ_n. If the eigenvectors are a complete and orthonormal set of vectors, show that B can be written as:

$$A = \sum_n \lambda_n |u_n\rangle\langle u_n| \,. \tag{1.124}$$

(Hint: the LHS and RHS of the above equation are both operators. Act with these operators on a generic ket $|\psi\rangle$ and show that you get the same result.)
Eq. 1.124 is called a *spectral decomposition* of B.

- Let A, B, C be three operators, show that:

$$[AB, C] = A\,[B, C] + [A, C]\,B \,. \tag{1.125}$$

1.5.6 More on X and P

Let X and P be the position and momentum operator, satisfying the canonical commutation relation

$$[X, P] = i \,, \tag{1.126}$$

show that:

$$[X^n, P] = inX^{n-1} \,, \tag{1.127}$$

$$[f(X), P] = i\frac{d}{dX} f(X) \,. \tag{1.128}$$

(Hint: use the Taylor expansion for f, and then apply the first equation.)

1.5.7 Parity operator

Let us consider a physical system whose state space is $\mathcal{E}$. The *parity operator* can be defined by its action on the eigenstates of the position operator:

$$\Pi|\mathbf{x}\rangle = |-\mathbf{x}\rangle \,. \tag{1.129}$$

Compute the matrix elements $\langle \mathbf{x}|\Pi|\mathbf{x}'\rangle$ of Π in the R representation.

The action of Π on a generic vector $|\psi\rangle$ is given by:

$$\Pi|\psi\rangle = \Pi \int d^3x \psi(\mathbf{x})|\mathbf{x}\rangle \,, \tag{1.130}$$

$$= \int d^3x' \psi(\mathbf{x}')|\mathbf{x}'\rangle \,. \tag{1.131}$$

Hence:

$$\langle \mathbf{x}|\Pi|\psi\rangle = \psi(-\mathbf{x}) \,. \tag{1.132}$$

1. Show that $\Pi^2 = 1$.
2. Using Eq. 1.132 show that $\langle \mathbf{x}|\Pi = \langle -\mathbf{x}|$, and deduce that Π is Hermitean.
3. Show that the eigenvalues of Π are limited to 1 and -1.
4. Let us define $P_\pm = \frac{1}{2}(1 \pm \Pi)$, show that $P_\pm^2 = P_\pm$, and that $P_\pm P_\mp = 0$.

Bibliography

[Cla05] Claude Cohen–Tannoudji, Bernard Diu, Frank Lanoë. *Quantum Mechanics*. John Wiley & Sons, Singapore, 2005.

[Hal58] P.R. Halmos. *Finite Dimensional Vector Spaces*. Van Nostrand, Princeton NJ, USA, 2nd edition, 1958.

[Tun85] Wu-Ki Tung. *Group theory in physics*. World Scientific, Singapore, 1985.