Chapter 4

Rotations

4.1 Geometrical rotations

Before discussing rotation operators acting the state space $\mathcal{E}$, we want to review some basic properties of geometrical rotations.

4.1.1 Rotations in two dimensions

In two dimensions rotations are uniquely defined by the angle of rotation. They preserve the length of a vector and the angle between vectors.

The image of a vector under a rotation by $\pi/3$ is represented in Fig. 4.1. Clearly the net result of

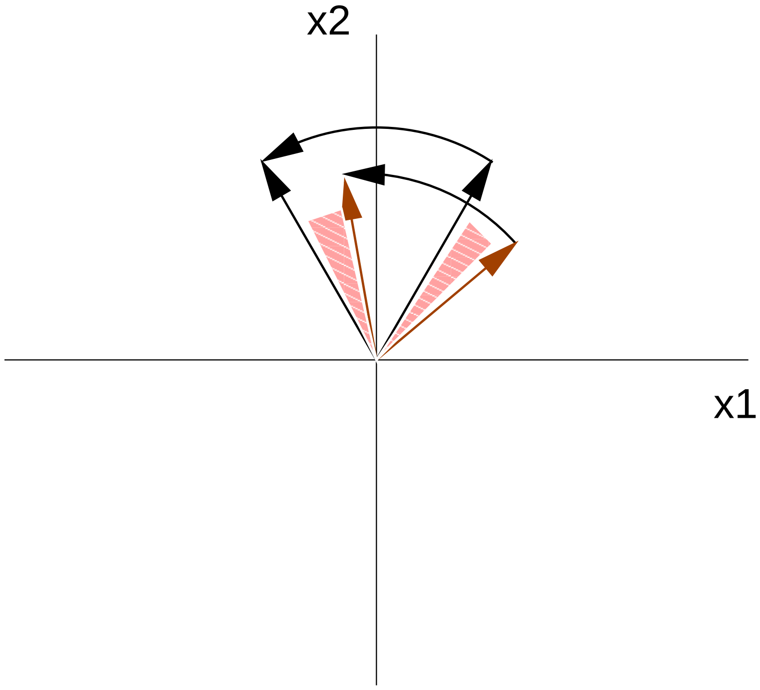

Figure 4.1: Rotation of vectors by $\pi/3$. You can see from the picture that the length of the vectors, and the angle between them are left unchanged.

two successive rotations is a rotation, the rotation by $\theta = 0$ is the identity, and any rotation can be undone by rotating in the opposite direction. The set of all two-dimensional rotations forms a group, called U(1). The elements of the group are labelled by the angle of the rotation $\theta \in [0, \pi)$. There is an infinite number of elements, denoted by a continuous parameter; groups where the elements are labelled by continuous parameters are called *continuous groups*. We will denote two–dimensional rotations by $\mathcal{R}(\theta)$. Note that the parameter labelling the rotations varies in a compact interval (the interval $[0, 2\pi)$ in this case). Groups with parameters varying over compact intervals are called *compact groups*.

The action of rotations on real vectors in two dimensions defines a representation of the group.

Given a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, a vector $\mathbf{r}$ is represented by two coordinates:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2\,. \tag{4.1}$$

The action of a rotation $\mathcal{R}(\theta)$ can be represented as 2×2 matrix:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.2}$$

Exercise 4.1.1 Check the formula above, then repeat it until you are sure you know it by heart!!

Intuitively two successive rotations by θ and ψ yield a rotation by $\theta + \psi$, and hence the group of two–dimensional rotations is Abelian.

Exercise 4.1.2 Using the two–dimensional representation of U(1) defined above, check that:

$$\mathcal{R}(\theta)\mathcal{R}(\theta') = \mathcal{R}(\theta + \theta')\,. \tag{4.3}$$

It is interesting to consider a one–dimensional complex representation of U(1). Given the coordinates (x_1, x_2) of a point in a two–dimensional space, we can define the complex number $z = x_1 + ix_2$. The transformation properties of z define a representation:

$$z \mapsto z' = e^{i\theta} z\,. \tag{4.4}$$

To each rotation $R(\theta)$ we can associate a single complex number $D(\theta) = e^{i\theta}$.

Exercise 4.1.3 Check that the following mappings define genuine representations of U(1):

$$\mathcal{R}(\theta) \mapsto D^{(n)}(\theta) = e^{in\theta}, \quad \forall n \in \mathbb{Z}\,. \tag{4.5}$$

What do we have for $n = 0$? What happens if $n \notin \mathbb{Z}$?

4.1.2 Rotations in three dimensions

Rotations in three dimensions are characterized by an axis (given by its unit vector $\mathbf{u}$, and the angle of rotation θ ($0 \leq \theta < 2\pi$). Hence a three–dimensional rotation is identified by three real parameters, and denotes by $\mathcal{R}_{\mathbf{u}}(\theta)$. The three real parameters can be chosen to be the components of a single vector:

$$\theta = \theta\mathbf{u} \tag{4.6}$$

whose length is given by the angle θ, and whose direction defines the axis of the rotation.

As for the two–dimensional case, the set of three–dimensional rotations constitutes a group, called SO(3). However SO(3) is not Abelian:

$$\mathcal{R}_{\mathbf{u}}(\theta)\mathcal{R}_{\mathbf{u}'}(\theta') \neq \mathcal{R}_{\mathbf{u}'}(\theta')\mathcal{R}_{\mathbf{u}}(\theta)\,. \tag{4.7}$$

Rotation around a given axis define subgroups of SO(3). Each of these subgroups is isomorphic to U(1).

Infinitesimal rotation Since rotations are identified by a continuous rotation angle, we can consider rotations by infinitesimally small angles.

The action of an infinitesimal rotation on a vector is given by:

$$\mathcal{R}_{\mathbf{u}}(d\theta)\mathbf{v} = \mathbf{v} + d\theta\mathbf{u}\times\mathbf{v}\,. \tag{4.8}$$

Exercise 4.1.4 Draw a plot to illustrate Eq. (4.8).

Every finite rotation can be decomposed as a product of infinitesimal ones:

$$\mathcal{R}_{\mathbf{u}}(\theta+d\theta) = \mathcal{R}_{\mathbf{u}}(\theta)\mathcal{R}_{\mathbf{u}}(d\theta) = \mathcal{R}_{\mathbf{u}}(d\theta)\mathcal{R}_{\mathbf{u}}(\theta)\,. \tag{4.9}$$

Exercise 4.1.5 Show that:

$$\mathcal{R}_{\mathbf{y}}(-d\theta')\mathcal{R}_{\mathbf{x}}(d\theta)\mathcal{R}_{\mathbf{y}}(d\theta')\mathcal{R}_{\mathbf{x}}(-d\theta) = \mathcal{R}_{\mathbf{z}}(d\theta d\theta')\,. \tag{4.10}$$

Before you perform the explicit calculation, can you explain why the result has to be proportional to $d\theta d\theta'$?

4.2 Rotations in state space: angular momentum

Let us consider a single particle in three-dimensional space. At any given time the state of the particle is described by a vector in a Hilbert space $|\psi\rangle \in \mathcal{E}$. The associated wave function is obtained by projecting the state vector on the basis of eigenfunctions of the position operator:

$$\psi(\mathbf{r}) = \langle \mathbf{r}|\psi\rangle\,. \tag{4.11}$$

We can now rotate the system by a rotation $\mathcal{R}$, such that:

$$\mathbf{r} \mapsto \mathbf{r}' = \mathcal{R}\mathbf{r}\,; \tag{4.12}$$

the state of the system after the rotation is described by a *different* vector $|\psi'\rangle \in \mathcal{E}$, and its associated wave function $\psi'(\mathbf{r}) = \langle \mathbf{r}|\psi'\rangle$. It is natural to assume that the value of the initial wave function at the point $\mathbf{r}$ will be rotated to the point $\mathbf{r}'$:

$$\psi'(\mathbf{r}') = \psi(\mathbf{r}) \Longleftrightarrow \psi'(\mathbf{r}') = \psi(\mathcal{R}^{-1}\mathbf{r}') \,. \tag{4.13}$$

Since the latter relation holds for all $\mathbf{r}'$, it can be rewritten as:

$$\psi'(\mathbf{r}) = \psi(\mathcal{R}^{-1}\mathbf{r}) \,. \tag{4.14}$$

We can define the operator R associated with the geometrical rotation $\mathcal{R}$ as the operator that associates the state $|\psi'\rangle$ to the state $|\psi\rangle$:

$$\begin{aligned} R : \mathcal{E} &\rightarrow \mathcal{E} \\ |\psi\rangle &\mapsto |\psi'\rangle = R|\psi\rangle \,. \end{aligned} \tag{4.15}$$

Eq. (4.14) can now be rewritten as:

$$\langle \mathbf{r}|R|\psi\rangle = \langle \mathcal{R}^{-1}\mathbf{r}|\psi\rangle \,. \tag{4.16}$$

The operator R is called a *rotation operator*.

Exercise 4.2.1 Prove that:

1. R is a linear operator;
2. R is unitary *(Hint: Consider the action of R on bras $\langle \mathbf{r}|$ and kets $|\mathbf{r}\rangle$)*;
3. the set of operators R defines a *representation* of the group of geometrical rotations.

For a small rotation angle $d\theta$, *e.g.* around the z axis, the rotation operator can be expanded at first order in $d\theta$:

$$R_z(d\theta) = 1 - i d\theta L_z + O(d\theta^2) \,; \tag{4.17}$$

the operator L_z is called the *generator* of rotations around the z axis. A finite rotation can then be written as:

$$R_z(\theta) = \exp\left(-i\theta L_z\right) \,. \tag{4.18}$$

The generators of rotations around the other axes L_x, L_y are defined in an analogous way.

Rotation operators in terms of angular momentum Let us assume the vector $\mathbf{r}$ is described by its coordinates (x, y, z) in a given basis, and let us consider the transformation of the wave function under a rotation by $d\theta$ around the $\mathbf{z}$ axis. According to the discussion in the previous Sections, we can write:

$$\mathcal{R}_z^{-1}(d\theta) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y d\theta \\ y - x d\theta \\ z \end{pmatrix} \,, \tag{4.19}$$

and therefore:

$$\psi'(x,y,z) = \psi(x + y d\theta, y - x d\theta, z). \tag{4.20}$$

Expanding at first order in $d\theta$ yields:

$$\psi'(x,y,z) = \psi(x,y,z) - i d\theta \left[x \frac{\partial}{i\partial y} - y \frac{\partial}{i\partial x} \right] \psi(x,y,z)\,. \tag{4.21}$$

Inside the square bracket you recognize the expression for the z component of the angular momentum in the R representation, $XP_y - YP_x$. We have shown a very important result: *the angular momentum operator in quantum mechanics is the generator of rotations in the space of physical states.* The angular momentum of a state describes the transformation properties of a given system under rotations. We will see several illustrations of this idea in the rest of the course.

From Eq. (4.21) we can easily derive:

$$\psi'(x,y,z) = \langle \mathbf{r} | \psi' \rangle = \langle \mathbf{r} | \left[1 - i d\theta L_z \right] | \psi \rangle\,; \tag{4.22}$$

since $\{|\mathbf{r}\rangle\}$ is a complete basis in $\mathcal{E}$, we deduce:

$$|\psi'\rangle = R_z(d\theta) = \left[1 - i d\theta L_z \right] |\psi\rangle\,. \tag{4.23}$$

The equation above is valid for arbitrary $|\psi\rangle$, and therefore we can write an identity between operators:

$$R_z(d\theta) = 1 - i d\theta L_z\,. \tag{4.24}$$

The image in the state space of the relation you proved in Eq. (4.10) can be written as:

$$\left[1 + i d\theta' L_y\right]\left[1 - i d\theta L_x\right]\left[1 - i d\theta' L_y\right]\left[1 + i d\theta L_x\right] = 1 - i d\theta d\theta' L_z\,; \tag{4.25}$$

expanding the left–hand side, and comparing the coefficients of the $d\theta d\theta'$ term we get the commutation relation of the components of angular momentum:

$$\left[L_x, L_y\right] = i L_z\,. \tag{4.26}$$

Note that the commutation relations of angular momentum operators are a consequence of the non–Abelian structure of the group of geometrical rotations.

The full set of commutation relations between generators can be computed by a similar method. They can be summarized as:

$$\left[L_i, L_j\right] = i \varepsilon_{ijk} L_k\,. \tag{4.27}$$

Exercise 4.2.2 Using the commutation relations above, show that

$$\left[\mathbf{L}^2, L_i\right] = 0\,, \tag{4.28}$$

where $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$.

The corresponding finite rotation operator is obtained as usual by exponentiating the generator:

$$R_z(\theta) = \exp\left[-i\theta L_z\right] \; ; \tag{4.29}$$

it can be generalized for a generic rotation around an axis $\mathbf{u}$:

$$R_{\mathbf{u}}(\theta) = \exp\left[-i\theta \mathbf{u}\cdot\mathbf{L}\right] \; . \tag{4.30}$$

Since the operators L_x, L_y, L_z do not commute:

$$R_{\mathbf{u}}(\theta) \neq \exp\left[-i\theta u_x L_x\right] \exp\left[-i\theta u_y L_y\right] \exp\left[-i\theta u_z L_z\right] \; . \tag{4.31}$$

Exercise 4.2.3 Knowing that the angular momentum is an observable, prove that the rotation operator R is unitary.

Finally let us consider again rotations around the z axis, and let us choose a basis in $\mathcal{E}$ composed of eigenvectors of L_z, $\{|m,\tau\rangle\}$. The variable τ indicates all the other indices that are needed to specify the vectors of the basis. Expanding a generic ket $|\psi\rangle$,

$$|\psi\rangle = \sum_{m,\tau} c_{m,\tau} |m,\tau\rangle \; , \tag{4.32}$$

where:

$$L_z|m,\tau\rangle = m|m,\tau\rangle \; . \tag{4.33}$$

Acting with a rotation operator on $|\psi\rangle$:

$$\begin{aligned} R_z(\theta)|\psi\rangle &= \sum_{m,\tau} c_{m,\tau} R_z(\theta)|m,\tau\rangle \\ &= \sum_{m,\tau} c_{m,\tau} e^{-im\theta} |m,\tau\rangle \; . \end{aligned} \tag{4.34}$$

You have seen in Quantum Mechanics lectures that the eigenvalues of the orbital angular momentum component L_z are integers, and therefore:

$$R_z(2\pi) = 1 \; . \tag{4.35}$$

The integers eigenvalues of L_z guarantee that the rotation operator corresponding to a rotation by 2π is the identity operator. This has to be the case if we are considering a single particle with no internal degrees of freedom: if the state of the particle is uniquely determined by its position in space, then upon a rotation by 2π the particle *has* to come back to its initial state. As we shall see later, this is no longer true if there are further degrees of freedom.

4.3 Commutation relations for a generic non-Abelian group.

The derivation we have just seen for the commutator of the generators of rotations can be generalized to any non-Abelian group.

Let us consider a generic continuous group G, whose elements are labelled by an n real parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$. We choose the parameters α in such a way that $g(\mathbf{0}) = e$.

Let D be a representation of the group. To each element $g(\alpha) \in G$ we associate a linear operator $D(\alpha)$ acting on some vector space $\mathcal{E}$. For an infinitesimal transformation $D(d\alpha)$ we can expand linearly in $d\alpha$:

$$D(d\alpha) = \mathbb{1} + \sum_k \alpha_k \left.\frac{\partial D}{\partial \alpha_k}\right|_0 + O(d\alpha_k^2)\,. \tag{4.36}$$

The generators T_k are defined by identifying the expression above with:

$$D(d\alpha) = \mathbb{1} - i \sum_k d\alpha_k T_k + O(d\alpha_k^2)\,, \tag{4.37}$$

i.e.

$$T_k = -i \left.\frac{\partial D}{\partial \alpha_k}\right|_0 . \tag{4.38}$$

For a finite transformation we have:

$$D(\alpha) = \lim_{N\to\infty} D(\alpha/N)^N = \lim_{N\to\infty} \left[1 - i\frac{\alpha_k T_k}{N}\right]^N = \exp[-i\alpha_k T_k]\,. \tag{4.39}$$

Exercise 4.3.1 The factor of i in the definition of the generators is a convention. It is particularly useful for physical purposes since unitary transformations D have Hermitean generators T_k. Prove this statement.

Consider now the group commutator:

$$D(\alpha)D(\beta)D(\alpha)^{-1}D(\beta)^{-1}\,, \tag{4.40}$$

this is the product of four elements of the group, and therefore it is a member of the group. Hence there must be a vector of n real parameters γ such that:

$$D(\alpha)D(\beta)D(\alpha)^{-1}D(\beta)^{-1} = D(\gamma)\,. \tag{4.41}$$

Clearly γ is a function of α, and β.

If the group is Abelian the matrices commute, and the group commutator reduces to the identity, *i.e.* $\gamma(\alpha, \beta) = 0$, while for a non-Abelian group we have:

$$\gamma(\alpha, \beta) \neq \mathbf{0}\,. \tag{4.42}$$

If we consider infinitesimal transformations, and expand $D(\alpha)$, $D(\beta)$, and $D(\gamma)$ at first order in their arguments, we find:

$$D(\gamma) = \mathbb{1} - i\gamma_k T_k = \mathbb{1} + \beta_l \alpha_m \left[T_l, T_m\right]\,. \tag{4.43}$$

Let us concentrate now on $\gamma(\alpha, \beta)$. The following properties can be easily proven:

1. $\gamma(0,0) = 0$;
2. if $\alpha = 0$, or $\beta = 0$, then $\gamma = 0$.

Therefore we conclude that γ must be a *quadratic* function of its arguments:

$$\gamma_t(\alpha, \beta) = -c_{rst}\alpha_r\beta_s \,, \tag{4.44}$$

where c_{rst} are real constants, and $r, s, t(= 1, \dots, n)$ are the indices labelling the parameters of the transformation.

Using Eq. (4.44), we obtain:

$$D(\gamma) = \mathbb{1} - ic_{rst}\alpha_r\beta_s T_t \,, \tag{4.45}$$

i.e.

$$[T_r, T_s] = ic_{rst}T_t \,. \tag{4.46}$$

The set of all real linear combinations of T_k is a vector space $\mathcal{G}$:

$$\mathcal{G} = \left\{ \sum_k c_k T_k, c_k \in \mathbb{R} \right\} \,. \tag{4.47}$$

The commutator $[,]$ defines a binary operation $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, such that:

$$[X+Y, Z] = [X,Z] + [Y,Z] \tag{4.48}$$
$$[X, Y+Z] = [X,Y] + [X,Z] \tag{4.49}$$
$$[\alpha X, \beta Y] = \alpha\beta[X,Y] \,. \tag{4.50}$$
$$\tag{4.51}$$

The vector space $\mathcal{G}$, equipped with the product law $[,]$ is called the *Lie algebra* of the group G. The coefficients c_{rst} which define the commutators of the generators are called *structure constants*.

Note that the structure constants were obtained starting from the parametrization $\gamma(\alpha, \beta)$ which are defined in a specific representations. It can be shown that they are actually independent of the representation.

Exercise 4.3.2 Write down explicitly the 3×3 matrix which represents a rotation by the angle θ around the z-axis in three dimensions. Expand its elements for infinitesimal θ and deduce the generator of rotations around z for this three-dimensional representation.

Exercise 4.3.3 Consider the set of 2×2 complex unitary matrices U, with $\det U = 1$. This set is a group under matrix multiplication called SU(2). Use the fact that $\det U = 1$ to prove that the generators are traceless. Since the matrices are unitary, the generators are also Hermitean. Check that the matrices $\sigma_i/2$, where σ_i are the Pauli matrices, are a basis for the Lie algebra of SU(2). Compute explicitly the commutation relations of the SU(2) generators $\sigma_i/2$, and check that they satisfy the same algebra as the SO(3) generators.

4.4 State space of two particles

Let us now consider a system composed of two spinless particles, which we denote by (1) and (2) respectively, and let us choose a basis $\left\{|\phi_i^1\rangle\right\}$ in the state space of particle (1), and a basis $\left\{|\phi_i^2\rangle\right\}$ in the space state of particle (2).

The state space of the composed system is given by the *tensor product*

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \,, \tag{4.52}$$

i.e. the vector space which is made of the linear combinations of the vectors:

$$|\phi_{ij}\rangle = |\phi_i^1\rangle|\phi_j^2\rangle \,. \tag{4.53}$$

Operators that refer to the particle (1) only act on the $|\phi_i^1\rangle$, and similarly for the operators that refer to particle (2), *e.g.* for a rotation operator:

$$R^1(\theta)|\phi_{ij}\rangle = \sum_k R^1(\theta)_i^k |\phi_k^1\rangle|\phi_j^2\rangle \,, \tag{4.54}$$

and similarly

$$R^2(\theta)|\phi_{ij}\rangle = \sum_l R^2(\theta)_j^l |\phi_i^1\rangle|\phi_l^2\rangle \,. \tag{4.55}$$

The generators of rotations are denoted $\mathbf{L}_1$, and $\mathbf{L}_2$ respectively.

Expanding a generic state of the two–particle system into the basis above

$$|\psi\rangle = \sum_{ij} c_{ij} |\phi_i^1\rangle|\phi_j^2\rangle \,, \tag{4.56}$$

it can be readily checked that it transforms according to the tensor product of the two representations R^1 and R^2:

$$|\psi\rangle \mapsto |\psi'\rangle = \left[R^1 \otimes R^2\right] |\psi\rangle \,. \tag{4.57}$$

Exercise 4.4.1 Check Eq. (4.57) by using the expansion of $|\psi\rangle$, and the rules above for the transformation properties of the basis vectors $|\phi_{ij}\rangle$.

The rotation operator can be written as:

$$R_{\mathbf{u}}^1(\theta) \otimes R_{\mathbf{u}}^2(\theta) = \exp\left[-i\theta \mathbf{L}_1 \cdot \mathbf{u}\right] \exp\left[-i\theta \mathbf{L}_2 \cdot \mathbf{u}\right] \,. \tag{4.58}$$

Since L_1 and L_2 commute, the rotation operator can be conveniently rewritten as:

$$R_{\mathbf{u}}^1(\theta) \otimes R_{\mathbf{u}}^2(\theta) = \exp\left[-i\theta (\mathbf{L}_1 + \mathbf{L}_2) \cdot \mathbf{u}\right] \,, \tag{4.59}$$

where the sum of the individual angular momenta $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ defines the *total angular momentum* of the composite system.

Intuitively you can argue that $\mathbf{L}_1$ and $\mathbf{L}_2$ commute because they relate to different particles within the composite system. This qualitative statement can be made more concrete using the formulae in Eqs. (4.54), and (4.55).

We shall see later the close relation between the Clebsch–Gordan series of a tensor product representation and the rules for the addition of angular momentum.

4.5 Rotation of an arbitrary system

So far we have discussed the relation between angular momentum and rotations starting from Eq. (4.14), which involved the wave function, *i.e.* the projection of the state vector onto eigenstates of position. However there is no need to refer to position eigenstates, and we can instead reason directly in terms of vectors in $\mathcal{E}$. We want to associate an operator R acting in $\mathcal{E}$ to each geometrical rotation $\mathcal{R}$. The operator R describes how the state vector changes under rotations. The product law of the group of geometrical rotations is preserved by the operators R (*i.e.* the mapping $\mathcal{R} \mapsto R$ is a homomorphism), but only locally: the product of two geometrical rotations, at least one of which being infinitesimal, is represented by the product of the corresponding operators R. However, the operator associated with geometrical rotations by 2π does not necessarily need to be mapped into the identity operator acting on $\mathcal{E}$.

The operator R_z corresponding to an infinitesimal geometrical rotation around the z-axis is:

$$R_z(d\theta) = 1 - id\theta J_z + \dots . \tag{4.60}$$

The equation above defines the generator of rotations along the z-axis, denoted J_z to distinguish the generic case from the orbital angular momentum discussed in previous Sections. The other components J_x, J_y are defined in a similar way. Using the same reasoning as above, we can show that the commutation relation for the J_i operators is the same as the one for the L_i that we already obtained.

The angular momentum of *any* quantum mechanical system is related to the corresponding rotation operators; the commutation relations amongst its components follow directly from this relation.

A finite rotation by θ around an axis $\mathbf{u}$ is written:

$$R_{\mathbf{u}}(\theta) = \exp\left(-i\theta \mathbf{u} \cdot \mathbf{J}\right) . \tag{4.61}$$

4.6 Rotation of operators

Given the transformation properties of physical states, we can easily derive the transformation properties of the operators that act on them. Let us consider a state $|\psi\rangle$ and its image under a rotation $|\psi'\rangle = R|\psi\rangle$. When acting on $|\psi\rangle$ with some operator O we obtain a state $|\phi\rangle = O|\psi\rangle$. Under the rotation R:

$$\begin{aligned} |\phi\rangle \mapsto R|\phi\rangle &= RO|\psi\rangle \\ &= (ROR^\dagger)(R|\psi\rangle) \\ &\equiv O'|\psi'\rangle ; \end{aligned} \tag{4.62}$$

i.e. the rotated operator is given by $O' = ROR^\dagger$.

Scalar operators A scalar operator is an operator which is invariant under rotations:

$$O' = O ; \tag{4.63}$$

if we consider infinitesimal rotations and use Eq. (4.60), the condition above translates into:

$$[O, \mathbf{J}] = 0 . \tag{4.64}$$

Several scalars appear in physical problems; *e.g.* the angular momentum squared $\mathbf{J}^2$ is a scalar.

Exercise 4.6.1 For a spinless particle ($\mathbf{J} = \mathbf{L}$), prove that $\mathbf{R}^2, \mathbf{P}^2, \mathbf{R} \cdot \mathbf{P}$ are invariant.

Vector operators A vector operator $\mathbf{V}$ is a set of three operators V_x, V_y, V_z which transform like the components of a geometric vector under rotations:

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \mapsto \begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = R \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} R^\dagger = \mathcal{R} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} , \tag{4.65}$$

where $\mathcal{R}$ is the 3×3 matrix describing the geometric rotation.

Exercise 4.6.2 Deduce the commutation relations of V_x with the three generators J_x, J_y, J_z.

Exercise 4.6.3 Consider a spinless particle and check explicitly that $\mathbf{R}$, and $\mathbf{P}$ are vector operators.
(Hint: the generator of rotations is given by $\mathbf{L} = \mathbf{R} \times \mathbf{P}$*, use the commutation relation between* $\mathbf{R}$ *and* $\mathbf{P}$ *to compute their commutation relations of* $\mathbf{L}$*.)*

Conservation of angular momentum A system is invariant under rotations if its Hamiltonian is invariant under all the elements of the group. For this to be true it is necessary and sufficient that the H commutes with all the generators of the rotation operators:

$$[H, \mathbf{J}] = 0\,; \tag{4.66}$$

this property can be rephrased by saying that the Hamiltonian is a scalar operator. It implies that the angular momentum is conserved.

The operators $H, \mathbf{J}^2, J_z$ commute with each other and therefore can be diagonalized simultaneously. We can choose a basis for the vector space of physical states made of eigenvalues of these three operators, $\{k, j, l\}$.

The degeneracy of the states and selection rules for the transition between these states are determined by their transformation properties under rotations, in complete analogy with the results we have established for finite groups in the previous Chapter.

4.7 Representations of SU(2)

We have seen in Sec. 4.3 that the SU(2) and SO(3) have the same algebra. The commutation relations for the generators are:

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \,. \tag{4.67}$$

Let us define $J^2 = \sum_i J_i^2$, it can be readily checked that $[J^2, J_i] = 0, \forall i = 1, 2, 3$. An operator that commutes with all generators of a Lie group is called a *Casimir operator*. When acting on the elements of a vector space that defines an irrep of the group, we obtain from Schur's lemma:

$$J^2 = \lambda \mathbb{1} \,, \tag{4.68}$$

and the eigenvalues of the Casimir operator can be used to classify the irreps.

Note that the J_i do not commute between themselves, and therefore cannot be simultaneously diagonalized. On the other hand a common basis can be found which diagonalizes both the commuting operators $\{J^2, J_3\}$. We shall denote $|j, m\rangle$ the elements of such a basis, j, m are two labels that identify the eigenvector; we shall see below that j is related to the eigenvalue of J^2, and m to the eigenvalue of J_3. The values that j, and m, can take are constrained by the structure of the group. The eigenvectors are normalized such that:

$$\langle j', m' | j, m\rangle = \delta_{jj'} \delta_{mm'} \,. \tag{4.69}$$

J_1, J_2 are used to build raising and lowering operators:

$$J_\pm = J_1 \pm i J_2, \quad (J_\pm)^\dagger = J_\mp \,. \tag{4.70}$$

The commutation relations in Eq. (4.67) imply:

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_- \tag{4.71}$$

$$[J_+, J_-] = 2J_3 \,. \tag{4.72}$$

Exercise 4.7.1 Prove that the Casimir operator J^2 can be rewritten as:

$$J^2 = J_3^2 - J_3 + J_+ J_- = J_3^2 + J_3 + J_- J_+ \,. \tag{4.73}$$

Let us now build the irreducible representations of SU(2). We identify m with the eigenvalue of J_3:

$$J_3 |j, m\rangle = m |j, m\rangle \,. \tag{4.74}$$

Exercise 4.7.2 Prove that $J_\pm$ are raising and lowering operators of m; *i.e.*

$$J_3 J_+ |j, m\rangle = (m + 1) J_+ |j, m\rangle \,, \tag{4.75}$$

$$J_3 J_- |j, m\rangle = (m - 1) J_- |j, m\rangle \,, \tag{4.76}$$

Note that the value of j is left unchanged by $J_\pm$, therefore the raising and lowering operators create new vectors that belong to the same irreducible representation identified by j.

Thus, by acting with J_+ we can build a tower of states with increasing values of m. If we want the representation to be finite–dimensional, we need this sequence to stop, *i.e.* we need a vector $|\psi\rangle$ such that $J_+|\psi\rangle = 0$. For a given value of j, we denote this vector $|j,j\rangle$:

$$J_3|j,j\rangle = j|j,j\rangle, \quad J_+|j,j\rangle = 0\,. \tag{4.77}$$

Clearly j is the highest value of the J_3 eigenvalue in the representation labelled by j. It is related to the eigenvalue of J^2 by Eq. (4.73):

$$J^2|j,j\rangle = j(j+1)|j,j\rangle\,, \tag{4.78}$$

i.e. $|j,j\rangle$ is an eigenstate of J^2 with eigenvalue $j(j+1)$. Since $[J^2, J_\pm] = 0$ we deduce that:

$$J^2|j,m\rangle = j(j+1)|j,m\rangle,\, \forall m\,. \tag{4.79}$$

Now we can start from the vector $|j,j\rangle$ and apply the lowering operator J_-. Clearly we generate states with decreasing values of $m = j, j-1, j-2, \ldots$. Once again if we want a finite–dimensional representation, this sequence must also stop, *i.e.* there exists a value l such that $J_-|j,l\rangle = 0$. From this condition we can redily obtain:

$$\begin{aligned}
&\langle j,l|J_+J_-|j,l\rangle = 0\\
&\Longrightarrow j(j+1) - l(l-1) = 0\\
&\Longrightarrow j = -l\,.
\end{aligned} \tag{4.80}$$

Hence the possible values for m in the representation labelled by j are $m = -j, -j+1, \ldots, j-1, j$. As a consequence, we obtain $j = n/2, \quad n \in \mathbb{N}$, *i.e.* the irreducible representations of SU(2) are labelled by semi–integer numbers.

Exercise 4.7.3 When acting with raising and lowering operators on $|j,m\rangle$ we obtain an eigenstate of J_3 with eigenvalue $m \pm 1$. However the vector is not normalized to one. Hence:

$$J_\pm|j,m\rangle = N_\pm(j,m)|j,m\pm 1\rangle\,. \tag{4.81}$$

Prove that the normalization is:

$$N_\pm(j,m) = \left[(j \mp m)(j \pm m + 1)\right]^{1/2}\,. \tag{4.82}$$

We have built a set of orthonormal vectors $\{|j,m\rangle, m = -j, -j+1, \ldots, j-1, j\}$. When acting with J_i on one of these vectors, a linear combination of $|j,m'\rangle$ vectors is obtained. The set of vectors is therefore the basis of a vector space which is left invariant by the rotation operators. This is precisely the definition of an irreducible representation. The representation is $(2j+1)$–dimensional.

$$\left(J_3^{(j)}\right)_{mm'} = \langle j,m|J_3|j,m'\rangle = m\delta_{mm'}\,, \tag{4.83}$$

$$\left(J_\pm^{(j)}\right)_{mm'} = \langle j,m|J_\pm|j,m'\rangle = N_\pm(j,m)\delta_{m,m'\pm 1}\,. \tag{4.84}$$

The equations above fully specify the representations of the algebra.

Representation of the group elements Group elements are obtained by exponentiating the generators, *i.e.*

$$D^{(j)}(\theta)_{mm'} = \langle j, m| e^{-i\theta\cdot\mathbf{J}} |j, m'\rangle\,. \tag{4.85}$$

Let us focus on rotations around the z-axis. Since J_3 is diagonal, the matrix corresponding to a finite rotation by an angle ϕ around the z-axis is easily computed:

$$d^{(j)}(\phi)_{mm'} = e^{im\phi}\delta_{mm'}\,, \tag{4.86}$$

and in particular:

$$d^{(j)}(2\pi) = (-)^{2m} = (-)^{2j} \tag{4.87}$$

If the particle does not have any internal degree of freedom, *i.e.* its state is completely specified by one complex wave function $\psi(\mathbf{x})$, then:

$$d^{(l)}(2\pi)\psi(\mathbf{x}) = \psi(\mathcal{R}^{-1}\mathbf{x}) = \psi(\mathbf{x})\,, \tag{4.88}$$

and therefore necessarily $d^{(l)}(2\pi) = 1$, and $j \in \mathbb{N}$. The representations of SO(3) are labelled by integer values j.

If the particle has internal degrees of freedom, its wave function depends on the position $\mathbf{x}$ *and* the value of the internal degrees of freedom, that we shall represent as a further variable a: $\psi = \psi(\mathbf{x}, a)$. The corresponding state vector in $\mathcal{E}$ is denoted $|\psi_a\rangle$. If the system is rotated in space the internal degrees can mix:

$$|\psi_a\rangle \mapsto U_{ab}(\theta)R(\theta)|\psi_b >\,, \tag{4.89}$$

where U acts on the internal degrees of freedom, and R is the rotation operator we discussed for the spinless case. The wave function transforms as:

$$\langle\mathbf{x}|\psi_a\rangle \mapsto \langle\mathbf{x}|U_{ab}R|\psi_b\rangle = \langle\mathcal{R}^{-1}\mathbf{x}|U_{ab}|\psi_b\rangle\,, \tag{4.90}$$

i.e. the state vector is transformed by the action of (i) the usual rotation operator $R = \exp\left[-i\theta\cdot\mathbf{L}\right]$, where $\mathbf{L}$ is the orbital angular momentum, and (ii) a further operator U which acts on the internal degrees of freedom (we shall call these internal degrees of freedom *spin*).

Exercise 4.7.4 Make sure you understand this paragraph!!

Combining the orbital part and the spin part of the rotation operator for infinitesimal rotations yields:

$$\begin{aligned}\left[UR\psi(\mathbf{x})\right]_a &= \left(\delta_{ab} - i\theta\cdot\mathbf{L}\delta_{ab} - i\theta\cdot\mathbf{S}_{ab}\right)\psi_b(\mathbf{x}) && (4.91)\\ &= \left(\mathbb{1} - i\theta\cdot\mathbf{J}\right)_{ab}\psi_b(\mathbf{x})\,, && (4.92)\end{aligned}$$

where $\mathbf{J} = \mathbf{L} + \mathbf{S}$, $[\mathbf{L}, \mathbf{S}] = 0$.

We can compute the character corresponding to a rotation by θ in the j irreducible representation. Remember that rotations by the same angle around different axes are conjugate elements in SO(3) and therefore have the same character. The computation is more easily performed for rotations around the z axis:

$$\begin{aligned}
\chi^{(j)}(\theta) &= \sum_{m=-j}^{j} \left(d^{(j)}(\theta)\right)_{mm} \\
&= \sum_{m=-j}^{j} e^{im\theta} \\
&= e^{-ij\theta} \sum_{m=0}^{2j} \left(e^{i\theta}\right)^m \\
&= \frac{\sin(j+1/2)\theta}{\sin\theta/2}\,,
\end{aligned} \tag{4.93}$$

which proves a result we had anticipated in Eq. (3.73).

For $l = 1$ we recover the three–dimensional vector represenation (*i.e.* the representation under which geometrical vectors transform). The character is given by $\chi^{(1)}(\theta) = 1 + 2\cos\theta$, as you can easily check from the explicit expression for the matrices $\mathcal{R}(\theta)$.

4.8 Product representation

The set of vectors $\{|j,m\rangle|j',m'\rangle = |jj',mm'\rangle\}$ is a basis of a $(2j+1)\times(2j'+1)$–dimensional vector space $\mathcal{E}^{jj'}$. The action of the rotation operators in $\mathcal{E}^{jj'}$ defines a $(2j+1)\times(2j'+1)$–dimensional product representation of SU(2). In a generic case such a representation is reducible, and characters are going to be useful to obtain a Clebsch–Gordan series for the product representation.

Using Eq. (4.93), we can prove the following identity:

$$\chi^{(j)}(\phi)\chi^{(j')}(\phi) = \chi^{(j+j')}(\phi) + \chi^{(j-1/2)}(\phi)\chi^{(j'-1/2)}(\phi)\,, \tag{4.94}$$

where ϕ indicates the rotation angle.

The last term in the RHS above is of the same form as the LHS, with the indices decreased by 1/2; *i.e.* Eq. (4.94) is a recursion relation for the product on the LHS. We can iterate the identity until one of the indices j, j' becomes $= 0$.

Without loss of generality, we can assume that $j > j'$, then:

$$\chi^{(j)}\chi^{(j')} = \chi^{(j+j')} + \chi^{(j+j'-1)} + \ldots + \chi^{(j-j')}\,; \tag{4.95}$$

i.e. we have the Clebsch–Gordan series:

$$D^{(j)} \otimes D^{(j')} = \bigoplus_{J=|j-j'|}^{j+j'} D^{(J)}\,. \tag{4.96}$$

The state $|jj',mm'\rangle$ can be decomposed into vectors that transform according to the irreducible representations of SU(2), which we label by J:

$$|jj',mm'\rangle = \sum_{J=|j-j'|}^{j+j'} \sum_{M=-J}^{J} |J,M\rangle\langle J,M|jj',mm'\rangle\,. \tag{4.97}$$

Let us define the total angular momentum $\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$, where $\mathbf{J}^{(1)}, \mathbf{J}^{(2)}$ are the generators of rotations in the subspaces labelled respectively by j and j'. Then the states $|J, M\rangle$ are eigenvectors of $\mathbf{J}^2$ and J_3 with eigenvalues $J(J+1)$ and M respectively.

Eq. (4.97) describes the rule for the addition of angular momenta. The coefficients of these expansion are the Clebsch–Gordan coefficients:

$$\begin{aligned} C(jj'J; mm'M) &= \langle J, M | jj', mm' \rangle \\ &= 0, \quad \text{unless } m + m' = M, |j - j'| \leq J \leq j + j' \,. \end{aligned} \tag{4.98}$$

Finally we can use the fact that the $|J, M\rangle$ are a complete set of orthnormal vectors to prove two orthogonality relations for the Clebsch–Gordan coefficients.

The first one uses the completeness:

$$\begin{aligned} \langle j_1 j_2, m_1 m_2 | j_1' j_2', m_1' m_2' \rangle &= \delta_{j_1 j_1'} \delta_{j_2 j_2'} \delta_{m_1 m_1'} \delta_{m_2 m_2'} \\ &= \sum_{JM} \langle j_1 j_2, m_1 m_2 | J, M \rangle \langle J, M | j_1' j_2', m_1' m_2' \rangle \\ &= \sum_{JM} C(j_1 j_2 J; m_1 m_2 M)^* C(j_1' j_2' J; m_1' m_2' M) \,. \end{aligned} \tag{4.99}$$

While the second one uses the orthnormality:

$$\begin{aligned} \delta J J' \delta_{MM'} &= \langle J, M | J, M \rangle \\ &= \sum_{jj', mm'} C(jj'J; mm'M) C(jj'J'; mm'M') /, . \end{aligned} \tag{4.100}$$

Eq. (4.100) can be used to invert the relation between $|J, M\rangle$ and $|jj', mm'\rangle$:

$$|J, M\rangle = \sum_{jj', mm'} C(jj'J; mm'M)^* |jj'; mm'\rangle \,. \tag{4.101}$$

Exercise 4.8.1 Consider the case $j_1 = j_2 = 1/2$. The states of each particle are vectors in two–dimensional vector spaces $\mathcal{E}_1$, $\mathcal{E}_2$. A basis for $\mathcal{E}_1$ is given by the states with $J_3 = \pm 1/2$, *i.e.* the states with spin up and down respectively, which we denote by $|\uparrow\rangle_1 = |1/2, 1/2\rangle$, and $|\downarrow\rangle_1 = |1/2, -1/2\rangle$ respectively. A basis for $\mathcal{E}_2$ can be constructed in a completely analogous way. The tensor space describing the states of the two particle system is the four–dimensional vector space spanned by the vectors:

$$\begin{array}{ll} |\uparrow\rangle_1 |\uparrow\rangle_2 = |1/2, 1/2; 1/2, 1/2\rangle & |\uparrow\rangle_1 |\downarrow\rangle_2 = |1/2, 1/2; 1/2, -1/2\rangle \\ |\downarrow\rangle_1 |\uparrow\rangle_2 = |1/2, 1/2; -1/2, 1/2\rangle & |\downarrow\rangle_1 |\downarrow\rangle_2 = |1/2, 1/2; -1/2, -1/2\rangle \,. \end{array}$$

From the discussion above the total angular momentum of the system of two particles of spin 1/2 can take the values $J = 0, 1$. Write the four states $|J = 0, J_3 = 0\rangle$, $|J = 1, J_3 = 1\rangle$, $|J = 1, J_3 = 0\rangle$, and $|J = 1, J_3 = -1\rangle$ as linear combinations of the four states listed above.

4.9 Wigner–Eckart theorem

Tensor operators We can now generalize the concept of tensor operator. A set of operators $\{T^{(j)}_m, m = -j, \ldots, j\}$ are called *tensor operators* if they transform according an irreducible representation of SU(2) under rotations:

$$T^{(j)}_m \mapsto R T^{(j)}_m R^{-1} = T^{(j)}_{m'} D^{(j)}_{m'm}(\theta)\,. \tag{4.102}$$

Exercise 4.9.1 By considering an infinitesimal rotation, prove that Eq. (4.102) is equivalent to:

$$\left[J_i, T^{(j)}_m\right] = T^{(j)}_{m'} \left(J^{(j)}_i\right)_{m'm}\,, \tag{4.103}$$

where $J^{(j)}_i$ is the matrix representing the generator J_i in the irreducible representation j.

We can use the transformation properties of operators and states do derive the selection rules that are induced by rotational symmetry.

Let $T^{(J)}_M$ be a tensor operator. The state $T^{(J)}_M|j,m\rangle$ transforms according to $D^{(J)} \otimes D^{(j)}$; such a state can be written as a linear combinations of the basis vectors $|Jj, Mm\rangle$, *i.e.*

$$T^{(J)}_M|j,m\rangle = t|Jj, Mm\rangle\,, \tag{4.104}$$

where t is a linear operator, invariant under rotations.

Theorem 4.9.1 The *Wigner–Eckart theorem* states that:

$$\langle j', m'|T^{(J)}_M|j,m\rangle = C(Jjj'; Mmm')\langle j'||T^{(J)}||j\rangle\,; \tag{4.105}$$

i.e. the matrix element is proportional to the Clebsch–Gordan coefficient. The proportionality factor $\langle j'||T^{(J)}||j\rangle$ only depends on J, j, and j', and is called a *reduced matrix element.*

4.10 Problems

4.10.1 More on generators

Under translations $\hat{T}(a)$ the position operator $\hat{x}$ transforms as $\hat{T}(a)\hat{x}\hat{T}^{-1}(a) = \hat{x} + a$. By writing $\hat{T} = e^{ia\hat{P}}$, where $\hat{P}$ is the generator of translations, deduce directly the canonical commutation relation

$$[\hat{x}, \hat{P}] = i.$$

4.10.2 Eigenfunctions of L_z

Show that the eigenfunctions $\psi^{(m)}(r, \theta, \phi)$ of $\hat{L}_z$ transform according to the irreducible representations of $SO(2)$. Use the orthogonality of the eigenfunctions to deduce appropriate orthogonality and completeness relations for the irreducible representations.

4.10.3 Rotations of vectors

The 3-dimensional real vector $\mathbf{r}'$ obtained by rotating the vector $\mathbf{r}$ through an angle θ about an axis in the direction of the unit vector $\mathbf{n}$ is given by

$$\mathbf{r}' = \mathbf{n}(\mathbf{n} \cdot \mathbf{r}) + \cos\theta[\mathbf{r} - \mathbf{n}(\mathbf{n} \cdot \mathbf{r})] + \sin\theta(\mathbf{n} \times \mathbf{r}).$$

1. Draw a plot to illustrate the above formula.
2. Use this to show that a general element of $SO(3)$ may be parameterised as

   $$R_{ij} = \cos\theta\,\delta_{ij} + (1 - \cos\theta)n_i n_j - \sin\theta \sum_{k=1}^{3} \varepsilon_{ijk} n_k$$

   What is the corresponding character?
3. Use this result to show that the infinitesimal generators of $SO(3)$ in the defining representation are

   $$(T_k)_{ij} = i\varepsilon_{ijk}.$$

4.10.4 Euclidean group

The Euclidean group in the plane, E_2, relates a point (x, y) to a point (x', y') in the plane by the transformation

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta + a \\ y' &= x\sin\theta + y\cos\theta + b \end{aligned}$$

where θ is an angle of rotation in the plane about the origin, and a, b are the components of a translation in the plane.

1. Show that the transformation may be represented as a 3×3 matrix which carries $(x, y, 1)$ to $(x', y', 1)$.

2. Denoting the infinitesimal generators in this representation by 3 by 3 matrices T_a, T_b and T_θ, show that they satisfy the Lie algebra

$$[T_a, T_b] = 0, \quad [T_\theta, T_a] = iT_b, \quad [T_\theta, T_b] = -iT_a.$$

3. Show that the three operators

$$\hat{T}_a = -i\frac{\partial}{\partial x}, \qquad \hat{T}_b = -i\frac{\partial}{\partial y}, \qquad \hat{T}_\theta = i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

also satisfy the same Lie algebra.

4.10.5 Killing form

This is a more abstract problem, you need to read carefully the notes on Lie algebras before you try to solve it.

Associated with a Lie algebra there is a *Killing form* defined in terms of the structure constants:

$$g_{pq} = g_{qp} \equiv \sum_{r,s} c_{prs}\, c_{qsr}$$

Show that for the Lie algebra $so(3)$, the Killing form is $g_{pq} = -2\delta_{pq}$. What is the Killing form for the Euclidean group E_2?

4.10.6 More on vector operators

(a) Show that if $\hat{V}_\pm \equiv \hat{V}_1 \pm i\hat{V}_2$, $\hat{V}_3$ are the three components of a vector operator, then

$$\left[\hat{J}^2, \hat{V}_q\right] = 0 \quad \left[\hat{J}_3, \hat{V}_\pm\right] = \pm\,\hat{V}_\pm \quad \text{and} \quad \left[\hat{J}_\pm, \hat{V}_\mp\right] = \pm 2\hat{V}_3, \quad \left[\hat{J}_\pm, \hat{V}_3\right] = \mp\hat{V}_\pm.$$

while all other commutators of $\hat{J}$ with $\hat{V}$ vanish.

(b) Given vector operators $\hat{x}_i$ and $\hat{P}_j$, such that $[\hat{x}_i, \hat{P}_j] = i\delta_{ij}$, show that the operator $\hat{x}\hat{P}$ is a reducible tensor operator, and decompose it into its three irreducible components.

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