Methods of Mathematical Physics

Exam style questions

1. Cauchy's integral formula states that if $f(z)$ is analytic within and on a closed contour C

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

where z_0 is within the contour C.

Consider a function $f(z)$ analytic in the upper half plane and on the real axis, and such that $|f(z)| \to 0$ as $|z| \to \infty$ for $0 \leq \arg z \leq \pi$. Show by using Cauchy's integral formula and an appropriate closed contour C that

$$f(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z_0}\, dx$$

where the integral is along the real axis . **[6]**

Now set $z_0 = x \pm i\epsilon$ and let z_0 approach the real axis. Show that

$$\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0 \mp i\epsilon}\, dx = \mathrm{P} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx \pm i\pi f(x_0)$$

[6]

where P indicates the Cauchy principal value.

Hence deduce

$$\frac{1}{x - x_0 \mp i\epsilon} = \mathrm{P}\frac{1}{x - x_0} \pm i\pi\delta(x - x_0)$$

and

$$f(x_0) = \frac{1}{\pi i} \mathrm{P} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx$$

[4]

Now consider a function f such that

$$f(-x) = f^*(x)\ .$$

Let $f(x_0) = u(x_0) + iv(x_0)$,where $u(x_0)$ and $v(x_0)$ are real functions, and show

$$u(x_0) = \frac{2}{\pi} \mathrm{P} \int_0^{\infty} \frac{xv(x)}{x^2 - x_0^2}\, dx$$

$$v(x_0) = -\frac{2}{\pi} \mathrm{P} \int_0^{\infty} \frac{x_0 u(x)}{x^2 - x_0^2}\, dx$$

[9]

2.

(a) The Gamma function may be defined as

$$\Gamma(x) = \int_0^\infty u^{x-1}\mathrm{e}^{-u}\,\mathrm{d}u \quad \text{where} \quad \mathrm{Re}[x] > 0\,.$$

Show that

$$\Gamma(1) = 1 \quad ; \quad \Gamma(1/2) = \sqrt{\pi} \quad ; \quad \Gamma(-1/2) = -2\sqrt{\pi}$$ **[5]**

Show that the Laplace transform of t^λ is given by

$$L[t^\lambda] = \frac{\Gamma(\lambda+1)}{s^{\lambda+1}}$$ **[3]**

(b) The temperature distribution in a semi-infinite rod ($x > 0$) obeys

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}$$

with boundary conditions

$$\begin{aligned} T(x,0) &= 0 \quad x > 0 \\ T(0,t) &= T_0 \end{aligned}$$

Show that the Laplace transform with repsect to t

$$F(x,s) = \int_0^\infty T(x,t)\mathrm{e}^{-st}\,\mathrm{d}t$$

obeys

$$\frac{\partial^2 F(x,s)}{\partial x^2} = s\frac{F(x,s)}{\kappa}\,.$$ **[3]**

Hence deduce

$$F(x,s) = \frac{T_0}{s}\exp\left(-(s/\kappa)^{1/2}x\right)\,.$$ **[4]**

(c) Identify the singularity of $F(x,s)$ which controls the large t behaviour of $T(x,t)$

Using the results of part a) compute the first three non-zero terms in the large t asymptotic expansion of $F(x,t)$. **[10]**

3. The generating function for the Bessel functions $J_n(x)$ is given by

$$G(x,t) = \sum_{n=-\infty}^{\infty} t^n J_n(x) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] .$$

From the symmetries $G(x,t) = G(x,-1/t)$, $G(x,t) = G(-x,1/t)$ deduce that

$$J_{-|n|}(x) = (-1)^n J_{|n|}(x) = J_{|n|}(-x) .$$ **[4]**

Use the product of generating functions

$$G(x+y,t) = G(x,t)G(y,t)$$

to derive

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x)J_{n-k}(y) .$$ **[4]**

By expanding the generating function show that

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(n+s)!\,s!}\left(\frac{x}{2}\right)^{n+2s} \qquad \text{for} \quad n \geq 0 .$$ **[6]**

Show that

$$R_n(x)J_n(x) = J_{n+1}(x) \quad \text{where} \quad R_n(x) = \left(\frac{n}{x} - \frac{\mathrm{d}}{\mathrm{d}x}\right)$$

$$L_n(x)J_n(x) = J_{n-1}(x) \quad \text{where} \quad L_n(x) = \left(\frac{n}{x} + \frac{\mathrm{d}}{\mathrm{d}x}\right)$$ **[6]**

Hence deduce the second order differential equation satisfied by $J_n(x)$. **[5]**

4. June 2002 Q1

Consider the partial differential equation

$$\nabla^2 u(\mathbf{x}) - m^2 u(\mathbf{x}) = \rho(\mathbf{x}) \tag{1}$$

in three dimensions.

a. Is this equation parabolic, elliptic, or hyperbolic? **[1]**

b. If $G(\mathbf{x}, \mathbf{x}')$ satisifies $(\nabla^2 - m^2)G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$ show that

$$G(\mathbf{x}, \mathbf{x}') = -(2\pi)^{-3} \int d^3\mathbf{k} \, \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{|\mathbf{k}|^2 + m^2}.$$

[4]

c. Show that

$$\int_0^\pi d\theta \, \sin\theta \, e^{-i\kappa r \cos\theta} = \frac{2 \sin \kappa r}{\kappa r}.$$

[5]

d. Show by contour integration that

$$\int_0^\infty d\kappa \, \frac{2\kappa \sin \kappa r}{(\kappa^2 + m^2)} = \pi e^{-mr}$$

[5]

e. Hence prove that

$$G(\mathbf{x}, \mathbf{x}') = -\frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x} - \mathbf{x}'|}.$$

[6]

f. Write down the solution to equation (1). **[4]**

5. June 2002 Q2

a. Find the Fourier transform of $f(x) = e^{-x^2/2V}$. **[5]**

b. Consider the function

$$g(x) = \sum_{n=-\infty}^{\infty} f(x + 2\pi n) = \sum_{n=-\infty}^{\infty} e^{-(x+2\pi n)^2/2V}; \tag{2}$$

show it is a periodic function, and find its period. **[5]**

c. Deduce that $g(x)$ has a Fourier series

$$g(x) = \sum_{k=-\infty}^{\infty} \tilde{g}_k e^{ikx}; \tag{3}$$

and show that $\tilde{g}_k = \sqrt{2\pi V} e^{-\frac{1}{2}k^2 V}$. **[8]**

d. Discuss how to use the expansions (2) and (3) to find approximate values for $g(x)$ for an arbitrary value of x in the cases where

(a) V is large, **[3]**

(b) V is small. **[4]**

6. June 2002 Q3

The Bessel function $J_0(x)$ may be defined by Schläfli's integral

$$J_0(x) = \frac{1}{2\pi i} \oint_{|t|=1} \frac{dt}{t} \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right].$$

For real $x \gg 1$ its asymptotic expansion may be found using the method of steepest descents.

a. Show that in the complex t plane the dominant contributions to the integral come from regions around $t_\pm = \pm i + 1/x + O(1/x^2)$. **[4]**

b. Show that the values of the integrand at the saddle points are

$$\exp\left\{\pm i\left[x - \frac{\pi}{2} + O\left(\frac{1}{x}\right)\right]\right\}.$$

[5]

c. Show that the steepest descents contour passes through the saddle points $t_\pm$ at angles $\alpha_+ = 3\pi/4$ and $\alpha_- = \pi/4$ to the positive real t axis. **[8]**

d. Show that the leading asymptotic behaviour is

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right).$$ **[8]**

[**N.B.** This question employs a slightly different approach to that of the lectures.]

7. Consider the integral

$$f_R(x) \equiv \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{-R}^{R} dz \, \frac{e^{ixz}}{z - i\varepsilon},$$

where $x > 0$.

a. Show that

$$f_R(x) = 1 - \frac{1}{2\pi} \int_0^\pi d\theta \, e^{ixRe^{i\theta}}.$$

[4]

b. Show that

$$|f_R(x) - 1| \le \frac{1}{2\pi} \int_0^\pi d\theta \, e^{-xR\sin\theta}.$$

[4]

c. Prove that $\sin\theta \ge 2\theta/\pi$ for $0 \le \theta \le \pi/2$. **[8]**

d. Show that

$$|f_R(x) - 1| \leq \frac{1}{2xR}.$$

[6]

e. Prove that

$$\lim_{R\to\infty} f_R(x) = \theta(x) \equiv \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

[3]

8. June 2001 Q3

Calculate the leading asymptotic behaviour of the Airy integral

$$\mathrm{Ai}(-x) = \frac{1}{\pi}\int_0^\infty \mathrm{d}w \, \cos(\frac{1}{3}w^3 - xw) \qquad x > 0\,,$$ **[25]**

as $x \to \infty$.

9. Let y_1 and y_2 be two solutions of Bessel's equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$.

a. If $W\Big(y_1(x), y_2(x)\Big) = W(x)$ is their Wronskian show that $x^2W' + xW = 0$, and hence that $W(x) = c/x$ where c is a constant. **[4]**

b. Using Frobenius' method one can show that the Bessel function

$$J_\nu(x) \equiv \sum_{r=0}^\infty \frac{(-1)^r}{r!\Gamma(\nu + r + 1)} \left(\frac{x}{2}\right)^{\nu+2r}$$

is a solution of of Bessel's equation. Manipulate this series to show that

$$J_{\nu-1}(x) \pm J_{\nu+1}(x) = \sum_{r=0}^\infty \frac{(-1)^r(\nu + r \mp r)}{r!\Gamma(\nu + r + 1)} \left(\frac{x}{2}\right)^{\nu+2r-1}.$$

and hence obtain the identities

$$\begin{aligned} J_{\nu-1}(x) + J_{\nu+1}(x) &= \frac{2\nu}{x} J_\nu(x) \\ J_{\nu-1}(x) - J_{\nu+1}(x) &= 2J'_\nu(x). \end{aligned}$$

[6]

c. Prove that

$$W\Big(J_\nu(x), J_{-\nu}(x)\Big) = \frac{x}{2\nu}\Big(J_{\nu-1}(x)J_{-\nu-1}(x) - J_{\nu+1}(x)J_{-\nu+1}(x)\Big).$$

[6]

d. Show that

$$W\Big(J_\nu(x), J_{-\nu}(x)\Big) = -\frac{2\sin\nu\pi}{\pi x}.$$

Hint: Consider the behaviour of $W(x)$ for small x, and recall that $B(1 + \nu, -\nu) = -\pi/\sin\nu\pi$. **[6]**

e. Under what circumstances are J_ν and $J_{-\nu}$ linearly independent? **[3]**

10. June 2003 Q1

(a) Find the position and nature of the singularities of the following differential equation.

$$x^2 y''(x) + x y'(x) - (x^2 + \nu^2) y(x) = 0 \,. \qquad (4)$$

[6]

(b) Set $y(x) = x^{-\frac{1}{2}} u(x)$ and show that u satisfies

$$u''(x) - \left[1 + \left(\frac{\nu^2}{x^2} - \frac{1}{4x^2}\right)\right] u(x) = 0 \,.$$

[5]

(c) Hence deduce that a solution which is bounded as $x \to \infty$ has the asymptotic behaviour

$$y(x) \sim A x^{-\frac{1}{2}} e^{-x} \quad \text{as} \quad x \to \infty \,.$$

[4]

(d) A solution of (4) is given by

$$y(x) = \frac{1}{2} \int_0^\infty dt\, t^{\nu-1} \exp\left[-\frac{x}{2}\left(t + \frac{1}{t}\right)\right] \,,$$

[you are not required to show this].

Using Laplace's method determine the asymptotic behaviour of this solution as $x \to \infty$. **[10]**

11. June 2003 Q2

A recurrence relation for Hermite polynomials is given by

$$H_{n+1}(x) = 2x\, H_n(x) - 2nH_{n-1}(x) \quad n \geq 0 \tag{1}$$

where $\quad H_0(x) = 1$ and $H_{-1}(x) = 0$.

Consider the generating function defined as

$$G(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)\,. \tag{2}$$

(a) Show that

$$\frac{\partial G(x,t)}{\partial t} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+1}(x)$$

$$tG(x,t) = \sum_{n=1}^{\infty} \frac{n\,t^n}{n!} H_{n-1}(x)\,.$$

[5]

(b) Hence deduce

$$G(x,t) = e^{-t^2+2tx}\,. \tag{3}$$

[4]

(c) From expression (3) for $G(x,t)$ establish the relation

$$\frac{dH_n(x)}{dx} = 2n\, H_{n-1}(x)\,. \tag{4}$$

[4]

(d) Using the definition (2) of $G(x,t)$ and the expression (3) deduce the integral representation

$$H_n(x) = \frac{n!}{2\pi i} \oint dt\, \frac{e^{-t^2+2tx}}{t^{n+1}}\,, \tag{5}$$

where the closed contour encircles the origin.

Verify that this expression satisfies the relation (4). **[5]**

(e) Use the integral representation (5) to demonstrate the orthogonality of the Hermite polynomials

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} H_m(x) H_n(x) = 2^n n! \sqrt{\pi}\, \delta_{m,n}\,.$$

[7]

Hint: Perform the x integral first then evaluate the subsequent two contour integrals.

12. June 2003 Q3

Consider the wave equation in three space dimensions and one time dimension for a system with periodic forcing

$$\nabla^2 u(\underline{x}, t) - \frac{1}{c^2}\frac{\partial^2 u(\underline{x}, t)}{\partial t^2} = f(\underline{x}) e^{-i\omega_0 t} . \tag{1}$$

(a) Show that $u(\underline{x}, t)$ may be written as

$$u(\underline{x}, t) = u(\underline{x}) e^{-i\omega_0 t} ,$$

where $u(\underline{x})$ satisfies

$$(\nabla^2 + k_0^2)\, u(\underline{x}) = f(\underline{x}) \quad \text{with} \quad k_0 = \frac{\omega_0}{c} .$$ **[2]**

(b) If $G(\underline{x},\, \underline{x}')$ satisfies

$$(\nabla^2 + k_0^2)\, G(\underline{x},\, \underline{x}') = \delta(\underline{x} - \underline{x}')$$

use $G(\underline{x},\, \underline{x}')$ to construct a solution to (1) . **[4]**

(c) Show by using Fourier transformation that

$$G(\underline{x},\, \underline{x}') = -\frac{1}{(2\pi)^3}\int d^3k\, \frac{e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}}{k^2 - k_0^2} \quad \text{where} \quad k = |\underline{k}| .$$ **[5]**

(d) Perform the angular integrals in $\underline{k}$-space to obtain

$$G(\underline{x},\, \underline{x}') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk\, \frac{k}{ir}\, \frac{e^{-ikr}}{k^2 - k_0^2} \quad \text{where} \quad r = |\underline{x} - \underline{x}'| .$$ **[5]**

(e) Perform this integral by contour integration. You should choose a contour that corresponds to outgoing waves $u \sim e^{i(k_0 r - \omega_0 t)}$, i.e., you should obtain the retarded Green function. **[9]**

13. June 2004 Q1

The Legendre Polynomials may be defined through the generating function

$$G(x, t) = \frac{1}{(1 - 2xt + t^2)^{1/2}}$$

where $G(x, t)$ is defined as

$$G(x, t) = \sum_{n=0}^{\infty} t^n P_n(x) .$$

(a) By taking derivatives of $G(x,t)$ with respect to t and x derive the following recursion relations

$$(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)xP_n(x)$$

$$P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) = P_n(x)\ . \qquad \textbf{[8]}$$

(b) Show that

$$\int_{-1}^{1} G^2(x,t)dx = \frac{1}{t}[\ln(1+t) - \ln(1-t)]\ .$$

By expanding the r.h.s. in powers of t and assuming

$$\int_{-1}^{1} dxP_n(x)P_m(x) = 0 \quad \text{for} \quad n \neq m$$

deduce

$$\int_{-1}^{1} P_n(x)P_{n'}(x)dx = \frac{2\delta_{n,n'}}{2n+1}\ . \qquad \textbf{[7]}$$

(c) A function $f(x)$ may be expressed as

$$f(x) = \sum_{k=0}^{\infty} a_k P_k(x)\ .$$

Use the result of part (b) to show that

$$a_n = (n+\frac{1}{2})\int_{-1}^{1} dxf(x)P_n(x)$$

and

$$\delta(x-x') = \frac{1}{2}\sum_{k=0}^{\infty}(2k+1)P_k(x)P_k(x')\ . \qquad \textbf{[5]}$$

(d) Show that $G(x,t)$ implies the following integral representation of $P_n(x)$

$$P_n(x) = \oint \frac{dt}{2\pi i}\,\frac{1}{t^{n+1}(1-2xt+t^2)^{1/2}}\ ,$$

where the closed contour encircles the origin. Specify the singularities of the integrand for the cases $|x| < 1$ and $|x| \geq 1$ and discuss the restrictions they impose on the radius of a circular contour. **[5]**

14. June 2004 Q2

(a) The Laplace transform of a function $f(t)$ is defined as

$$L[f] = \int_0^\infty f(t)e^{-st}dt\ , \quad \text{where} \quad s > 0\ .$$

Show that

$$\begin{aligned} L[tf] &= -\frac{\partial}{\partial s}L[f]\ , \\ L\left[\frac{df}{dt}\right] &= -f(0) + sL[f] \end{aligned}$$ **[3]**

and

$$L[t^\lambda] = \frac{\Gamma(\lambda+1)}{s^{\lambda+1}} \quad \text{where} \quad \Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du\ .$$ **[2]**

(b) Consider the differential equation

$$t^2\frac{d^2y(t)}{dt^2} + t\frac{dy(t)}{dt} - (1+t^2)y(t) = 0\ .$$

Let $g(s)$ denote the Laplace transform of a solution of this equation which is bounded as $t \to \infty$.

Show that $g(s)$ obeys

$$(s^2-1)g''(s) + 3sg'(s) = 0\ .$$ **[5]**

Integrate this equation to obtain

$$g'(s) = \frac{A}{(s^2-1)^{3/2}}\ ,$$ **[3]**

where A is a constant.

Assuming the inversion integral for Laplace transforms, deduce that

$$y(t) = -\frac{A}{t}\int_C \frac{ds}{2\pi i}\ \frac{e^{st}}{(s^2-1)^{3/2}}\ ,$$

where you should specify the contour C. **[2]**

(c) Sketch the analytic structure of the integrand and identify the singularity that controls the large t behaviour of $y(t)$. **[3]**

Compute the first two non-zero terms in the large t asymptotic expansion of $y(t)$. **[7]**

15. June 2004 Q3

Consider the Green function $G(\underline{x}, t)$ in three space dimensions and one time dimension defined by

$$\left[\nabla^2 - \frac{\partial^2}{\partial t^2} - m^2\right] G(\underline{x}, t) = \delta(\underline{x})\delta(t) \ .$$

Show by Fourier transformation that

$$G(\underline{x}, t) = \frac{1}{(2\pi)^4} \int d^3k \int d\omega \, \frac{e^{i\underline{k}.\underline{x} - i\omega t}}{(\omega^2 - k^2 - m^2)} \quad \text{where} \quad k = |\underline{k}| \ .$$ **[5]**

Perform the angular integrations in $\underline{k}$-space to obtain

$$G(\underline{x}, t) = \frac{1}{(2\pi)^3 r} \int_{-\infty}^{\infty} dk \, \frac{k}{i} \, e^{ikr} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2 - \Omega^2} \quad \text{where} \quad \Omega = \sqrt{k^2 + m^2}$$

and $r = |\underline{x}|$. **[5]**

Perform the ω integral by contour integration, choosing the contour to give

$$\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega t}}{\omega^2 - \Omega^2} = -2\pi \frac{\sin(\Omega t)}{\Omega} \, \theta(t) \ ,$$

where $\theta(t)$ is the usual Heaviside step function. **[7]**

Use the following results (which you are not required to show) for Bessel functions J_0, J_1

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ikr} \frac{\sin(\Omega t)}{\Omega} = \frac{1}{2} J_0\left(m\sqrt{t^2 - r^2}\,\right) \theta(t^2 - r^2)$$

$$\begin{aligned} J_0'(y) &= -J_1(y) \\ J_0(0) &= 1 \end{aligned}$$

to deduce that

$$G(\underline{x}, t) = \left[\frac{m}{4\pi\sqrt{t^2 - r^2}} \, J_1\left(m\sqrt{t^2 - r^2}\,\right) \theta(t - r) - \frac{1}{4\pi r} \, \delta(r - t)\right] \theta(t) \ .$$ **[5]**

Interpret the two terms in $G(\underline{x}, t)$ identifying the case $m = 0$. **[3]**