Section 1: Introduction; Revision of Infinite Series; Asymptotic Expansions

This course will develop and extend topics that you may have met before. To some extent we will be redoing some topics (e.g. Fourier and Laplace transforms, Green functions) now that you have a knowledge of complex analysis. However we shall also extend your mathematical toolbox to include techniques that you won't have met before e.g. asymptotic expansions, saddle-point methods...

The first two lectures will review the analysis of infinite series and complex analysis respectively. Although for some this will mainly be revision, in both lectures there will be material that will be new to everyone. Varying amounts of rigour are exposed in the lectures (with a bit more in the lecture notes) to give you a flavour for solid mathematical argument. However the overiding aim of the course is to get you fluent in calculations so the important thing is for you to go through the worked examples and attempt the tutorial questions.

1. 1. Infinite series

Infinite series are ubiquitous in physics and arise from e.g.

- expansions of functions (Taylor series)
- perturbation theory
- expansion of integrals

One is particularly (but as we shall see not exclusively) interested in series which converge

Definition of convergence

$\sum_{n=1}^{\infty} a_n$ converges to S if $\forall \epsilon > 0\ \exists N_0\ \ s.t.\ \left|S - \sum_{n=1}^{N} a_n\right| < \epsilon \quad \forall N > N_0$

This a rigorous statement that if we add more terms to our series we get closer and closer to the finite value S i.e. the sequence of partial sums converges.

A series is *absolutely* convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

A well known example of an (absolutely) convergent series is $1 + x + x^2 \ldots = 1/(1-x)$ when $|x| < 1$; the series diverges if $|x| \geq 1$. (*Exercise:* show the sequence of partial sums $\sum_{n=1}^{N} x^n$ obeys the convergence criterion above)

Ratio test for convergence

$$\boxed{\text{if } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \text{ then the series } \sum_{n=1}^{\infty} a_n \text{ converges absolutely.}}$$

To understand this we should define the limit of a sequence $z_1, z_2, \ldots z_n$.

We say $\lim\limits_{n\to\infty} z_n = l$ if $\forall \epsilon > 0\ \exists n_0\ \ s.t.\ |z_n - l| < \epsilon\ \forall n > n_0$
i.e. we get arbitrarily close to the limiting value by considering later and later terms in the sequence.

Proof of ratio test: Consider

$$\sum_{n=N}^{\infty} a_n \le \sum_{n=N}^{\infty} |a_n| = |a_N| \left\{ 1 + \left|\frac{a_{N+1}}{a_N}\right| + \left|\frac{a_{N+2}}{a_{N+1}}\right| \left|\frac{a_{N+1}}{a_N}\right| + + \left|\frac{a_{N+3}}{a_{N+2}}\right| \left|\frac{a_{N+2}}{a_{N+1}}\right| \left|\frac{a_{N+1}}{a_N}\right| + \cdots \right.$$

Now, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = K < 1$, this means that for any given $\epsilon < 1 - K$ we can find n_0 s.t. $\left|\frac{a_{n+1}}{a_n}\right| \le K + \epsilon < 1$ for $n > n_0$.

Then for $N > n_0$, $\sum_{n=N}^{\infty} a_n \le |a_N| \sum_{n=0}^{\infty} (K+\epsilon)^n$, which converges. Thus

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$$

converges since the first sum contains only a finite number of terms and so is finite and we have shown that the second sum converges.

The idea of this proof is basically to compare the general sum with a series (here the geometric series) that we know is convergent.

Clearly if if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ then because the terms get bigger the series will not converge, so the ratio test is inconclusive only in the case $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Comparison with an integral: If $f(x)$ is montonically decreasing

$$\sum_{n=b}^{\infty} f(n) \le \int_{b-1}^{\infty} f(x)dx$$

(convince yourself by drawing a sketch).
Thus convergence of the integral $\Rightarrow$ convergence of the series.

Similarly

$$\sum_{n=b}^{\infty} f(n) \ge \int_{b}^{\infty} f(x)dx$$

(again convince yourself) thus divergence of the integral $\Rightarrow$ divergence of the series.

Thus we can compare with the appropriate integral to determine convergence of a series

Example: Riemann zeta function. This function is defined by

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(it appears in theory of Bose condensation and is of also of great importance in studying the distribution of prime numbers!)

The ratio test yields $\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{n}\right)^{-s} \sim 1 - \frac{s}{n} \to 1$ as $n \to \infty$ which is inconclusive.

However

$$\int \frac{dx}{x^s} = -\frac{1}{s-1} \frac{1}{x^{s-1}}$$

converges as $x \to \infty$ when $s > 1$ thus $\zeta(s)$ converges for $s > 1$.

Thus we can improve the ratio test to

if $\left|\frac{a_{n+1}}{a_n}\right| \to 1 - \frac{s}{n}$, where $s > 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
(Proof is analgous to proof of ratio test given above.)

1. 2. Techniques for evaluating series

There are a few series that you should know by heart (e.g. geometric series, expansion of e^x, binomial expansion ...). However other series can often be related to the few key ones you choose to memorise e.g.

Example 1.

$$\begin{aligned} \frac{d}{dx}\ln(1+x) &= \frac{1}{1+x} = 1 - x + x^2 \cdots \quad \text{(geometric series)} \\ \Rightarrow \ln(1+x) &= C + x - x^2/2 + x^3/3 \end{aligned}$$

and we determine the constant of integration as $C = 0$ through the condition $\ln(1) = 0$.

Example 2. $\quad S = 1 + 2x + 3x^2 \cdots = \frac{d}{dx}\sum_{n=1}^{\infty} x^n = \frac{d}{dx}\left[\frac{x}{1-x}\right] = \frac{1}{(1-x)^2}$

Example 3. $\quad S = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

Consider $f(x) = \sum_{n=1}^{\infty} \frac{n x^{n+1}}{(n+1)!}$. Now $f'(x) = xe^x$ and integrating by parts yields $f(x) = xe^x - e^x + 1$. Thus $S = f(1) = 1$.

Other methods for evaluating infinite series include using contour integrals (which we will revise next lecture), Fourier series and the 'telescoping trick' (see tutorial).

1. 3. Asymptotic series

Asymptotic series occur widely in physics. Before defining what they are we start with an example.

Consider the integral

$$G(x) = \int_0^{\infty} \frac{e^{-xt}}{1+t}\, dt \quad \text{convergent for } x > 0\ .$$

For $x \gg 0$ this is clearly dominated by small t (because of e^{-xt}) so let us expand

$$G(x) = \int_0^{\infty} e^{-xt} \sum_{n=0}^{\infty} (-t)^n dt = \sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^{n+1}} \quad \text{(integration by parts)}$$

and we have a nice looking series that we would hope to use for large x. But the ratio test fails i.e. $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{x} \to \infty$ as $n \to \infty$ so the series diverges.

However if we evaluate the first few terms (say four) of the series for $x = 10$ we find $G(x) \simeq 0.1 - 0.01 + 0.002 - 0.0006 = 0.0914$ which is very close to the correct value 0.09156 (the values of this integral are tabulated). So it appears that the series is actually useful!

Now consider the 'remainder' i.e. difference between $G(x)$ and the n^{th} 'partial sum' g_n

$$g_n = \sum_{m=0}^{n-1} (-1)^m \frac{m!}{x^{m+1}} = \int_0^\infty e^{-xt} \sum_{m=0}^{n-1} (-t)^m dt$$

Noting that

$$1 - t + t^2 \ldots + (-1)^{n-1} t^{n-1} = \frac{1-(-1)^n t^n}{1+t} \Rightarrow \frac{1}{1+t} = 1 - t + t^2 \ldots + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t}$$

we find

$$\begin{aligned} G(x) - g_n(x) &= \int_0^\infty e^{-xt} \frac{(-t)^n}{1+t} dt \\ \Rightarrow |G(x) - g_n(x)| &< \int_0^\infty t^n e^{-xt} dt = \frac{n!}{x^{n+1}} \end{aligned}$$

Thus the magnitude of the remainder is smaller than the first neglected term $(-1)^n \frac{n!}{x^{n+1}}$.

So although the series is ultimately divergent (the terms start to increase in magnitude when $|a_{n+1}/a_n| > 1$ i.e. $n+1 > x$) the first few terms which decrease in magnitude furnish a sequence of approximations of the integral for large x.

Definition of an asymptotic series

First let us define what we mean by 'of the order of' symbol O.

If $\left|\frac{a_n}{z_n}\right| < K$ (independent of n) for n sufficiently large then $a_n = O(z_n)$.

e.g. $\dfrac{15n+19}{1+n^3} = O(1/n^2)$.

Let $f(z) = a_0 + \dfrac{a_1}{z} + \cdots + \dfrac{a_{n-1}}{z^{n-1}} + R_n(z)$.

Suppose that the remainder $R_n(z) = O(z^{-n})$ then

$S = \sum a_s z^{-s}$ is an asymptotic series (or asymptotic expansion) for $f(z)$ i.e.

$$z^n \left\{ f(z) - \sum_{s=0}^{n} \frac{a_s}{z^s} \right\} \to 0 \text{ as } z \to \infty$$

It is important to remember that

An asymptotic series approaches $f(z)$ as $z \to \infty$ for a *given* n whereas a convergent series approaches $f(z)$ as $n \to \infty$ for a *given* z

Some further notes on asymptotic series

- Usually they are alternating series; the partial sums at first approach the function but then start to increase in size and oscillate wildly
- The best estimate for the function for a prescribed x is to include terms up to the minimum magnitude term in the series e.g. for the example $G(x)$ above one should keep terms up to n = integer part of x
- A given series can be the asymptotic series for several different functions
- It is meaningless to ask if a series is an asymptotic series without having a function to compare it to.