Section 8: Power Series Solutions of ODEs

In the last lecture we saw that for 2nd order linear differential equations knowledge of one solution of the homogeneous equation suffices to provide the general solution to the inhomogeneous equation. In this lecture we study how to obtain systematically a solution of the homogeneous equation.

8. 1. Classification of Singularities

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{1}$$

1. If $p(x_0)$ and $q(x_0)$ are finite then x_0 is an **ordinary** point.
2. If $p(x)$ and/or $q(x_0) \to \infty$ as $x \to x_0$ then x_0 is a **singular point**.
   regular if $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are finite as $x \to x_0$
   irregular (or **essential**) if $(x - x_0)p(x)$ and/or $(x - x_0)^2 q(x) \to \infty$ as $x \to x_0$.

For finite x_0 we determine the nature of singularities by inspection.

For the 'point at infinity' let $z = \dfrac{1}{x}$ and study the behaviour as $z \to 0$

$$\frac{d}{dx} = \frac{dz}{dx}\frac{d}{dz} = -z^2\frac{d}{dz}$$

$$\frac{d^2}{dx^2} = \frac{dz}{dx}\frac{d}{dz}\left[-z^2\frac{d}{dz}\right] = 2z^3\frac{d}{dz} + z^4\frac{d^2}{dz^2}$$

Thus (1) becomes letting $y(1/z) = Y(z)$

$$Y''(z) + \left[\frac{2}{z} - \frac{p(1/z)}{z^2}\right]Y'(z) + \frac{q(1/z)}{z^4}Y(z) = 0 \tag{2}$$

Example: SHM $\quad y''(x) + \omega^2 y(x) = 0$

Clearly there are no singularities at finite x

For $x \to \infty$ consider

$$Y''(z) + \frac{2}{z}Y'(z) + \frac{\omega^2}{z^4}Y(z) = 0$$

Which has an essential singularity at $z = 0$ ($x = \infty$)

Example: Legendre's equation

$$(1 - x^2)y''(x) - 2xy'(x) + \alpha y(x) = 0$$

Rewrite in conventional form as

$$y''(x) - \frac{2x}{1 - x^2}y'(x) + \frac{\alpha}{1 - x^2}y(x) = 0$$

which has regular singular points at $x = \pm 1$ since both $(x \mp 1)p(x)$ and $(x \mp 1)^2 q(x)$ are finite as $x \to \pm 1$.

In terms of $z = 1/x$ and $Y(z) = y(1/z)$ we have $Y''(z) + \dfrac{2z}{z^2-1}Y'(z) + \dfrac{\alpha}{z^2(z^2-1)}Y(z) = 0$ which has a regular singular point at $z = 0$ ($x = \infty$).

8. 2. Taylor expansion about ordinary point

For an ordinary point we may assume a Taylor expansion

$$y(x) = \sum_{m=0}^{\infty} c_m (x - x_0)^m \tag{3}$$

and this will give a convergent series with radius of convergence as least as large as the distance to the nearest singular point of (1).

Example: Legendre's equation

$$(1 - x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0$$

Inserting (3) with $x_0 = 0$ yields

$$(1 - x^2)\sum_{m=0} m(m-1)c_m x^{m-2} - 2x\sum_{m=0} mc_m x^{m-1} + n(n+1)\sum_{m=0} c_m x^m$$

Multiplying out and adjusting the m index in the first term yields

$$\sum_{m=0}^{\infty} x^m \left[(m+2)(m+1)c_{m+2} - m(m-1)c_m - 2mc_m + n(n+1)c_m\right]$$

Thus setting the coefficient of each power of x to zero

$$c_{m+2} = \frac{m(m+1) - n(n+1)}{(m+2)(m+1)}c_m = \frac{(m-n)(m+n+1)}{(m+2)(m+1)}c_m$$

Note that this is a second order recursion (relating c_{m+2} to c_m) thus there are two undetermined constants c_0 and c_1 giving two independent series

$$\begin{aligned} y(x) &= c_0\left[1 - n(n+1)\frac{x^2}{2!} + n(n+1)(n-2)(n+3)\frac{x^4}{4!} + \ldots\right] \\ &\quad + c_1\left[x - (n-1)(n+2)\frac{x^3}{3!} + (n-1)(n+2)(n-3)(n+4)\frac{x^5}{5!} + \ldots\right] \end{aligned}$$

At $x = \pm 1$ one can show by the ratio test that the series diverge. But if n is an even (odd) integer the even (odd series) terminates at x^n, these give the Legendre Polynomials.

8. 3. Frobenius expansion about regular singular point

For a *regular* singular point a straight Taylor expansion (3) fails. Instead one should try a Frobenius Series, which looks like

$$y(x) = (x - x_0)^{\alpha}\sum_{m=0}^{\infty} c_m (x - x_0)^m \qquad c_0 \neq 0 \tag{4}$$

i.e. $y(x) = (x - x_0)^{\alpha}A(x)$ where $A(x)$ is analytic at x_0 (i.e. can be expressed as a Taylor series). α is determined from the 'indicial equation' (see below) which is a quadratic generally giving two values α_1, α_2.

Note that if α in (4) is noninteger or a negative integer then $y(x)$ is *nonanalytic*. The case $\alpha = 0$ or positive integer reduces to a Taylor series.

One can prove (but it is a bit tedious so not here!) the following results often referred to as 'Fuch's theorem':

At a regular singular point there is *at least* one solution of Frobenius form

$$y_1(x) = (x - x_0)^\alpha A(x) \ .$$

1. If $\alpha_1 - \alpha_2 \neq 0$ or an integer there will be two independent solutions of Frobenius form
2. Otherwise the second solution will be of the form

   $$y_2(x) = \ln(x - x_0) y_1(x) + C(x)(x - x_0)^\beta \tag{5}$$

   (although exceptionally two Frobenius solutions do exist).

Example: Bessel's equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \tag{6}$$

There is a regular singular point at $x = 0$ so substitute $y = x^\alpha \sum_{m=0}^{\infty} x^m$:

$$\sum_{m=0} (\alpha + m)(\alpha + m - 1) c_m x^{\alpha+m} + \sum_{m=0} (\alpha + m) c_m x^{\alpha+m} - n^2 \sum_{m=0} c_m x^{\alpha+m} + \sum_{m=0} c_m x^{\alpha+m+2} = 0$$

The last sum begins at $x^{\alpha+2}$ whereas the first three begin at x^α. Thus setting coefficients of $x^{\alpha+s}$ to zero leads to

$$\begin{array}{lrcll}
\textbf{`indicial equation'} & c_0 \left[\alpha^2 - n^2\right] & = & 0 & \Rightarrow \alpha = \pm m \\
 & c_1 \left[(\alpha+1)^2 - n^2\right] & = & 0 & \Rightarrow c_1 = 0 \qquad (\text{unless} \quad n = 1/2) \\
\text{for} \quad m > 1 & c_m \left[(\alpha+m)^2 - n^2\right] + c_{m-2} & = & 0 &
\end{array}$$

The final equation gives $c_m = 0$ for m odd and for m even (replacing m by $m + 2$)

$$c_{m+2} = -\frac{c_m}{(m+2)(m+2+2\alpha)} \quad \text{where} \quad \alpha = \pm n \tag{7}$$

Now for n not an integer we actually get two independent solutions corresponding to $\alpha = \pm n$. But for n an integer we need to choose either $\alpha = n$ (n +ve) or $\alpha = -n$ (n -ve) otherwise the rhs will diverge for $m + 2 \mp 2n = 0$. Thus we only get one independent solution.

To illustrate what happens we consider the case n a +ve integer. Choosing $\alpha = n$ we iterate (7) and find

$$c_2 = -\frac{1}{2.2.(1+n)} c_0 \qquad c_4 = -\frac{1}{4.2.(2+n)} c_2 \qquad c_6 = -\frac{1}{6.2.(3+n)} c_4$$

Thus

$$c_{2p} = (-1)^p \frac{n!}{2^{2p} p! (n+p)!} c_0$$

and we have

$$y_1(x) = J_n(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,(p+n)!} \left(\frac{x}{2}\right)^{n+2p}$$

where we have chosen $c_0 = \dfrac{1}{2^n\, n!}$. $J_n(x)$ are Bessel functions of the first kind.

Let us find a second solution: take $n = 0$ for simplicity. Then

$$y_1(x) = J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - O(x^6)$$

We will use the Wronskian (lecture 6) to construct a second solution. From (6) $p(x) = 1/x$ so

$$W(x) = \exp - \int^x x^{-1} = \exp - \ln x = x^{-1}$$

and

$$\begin{aligned}
y_2(x) &= y_1(x) \int^x \frac{1}{t}\,\frac{1}{y_1^2(t)} dt \\
&= J_0(x) \int^x \frac{1}{t}\left[1 - \frac{t^2}{4} + \frac{t^4}{64} + \cdots\right]^{-2} dt \\
&= J_0(x) \int^x \frac{1}{t}\left[1 + \frac{t^2}{2} + \frac{5t^4}{32} + \cdots\right] dt \\
&= J_0(x) \left\{\ln x + \frac{x^2}{4} + \frac{5x^4}{128} + \cdots\right\}
\end{aligned}$$

and we see this is of the form (5).

Actually although this gives a perfectly acceptable second solution of Bessel's equation for integer n, other methods are usually employed—see *Arfken* or *RHB* for details.

8. 4. Expansion about an irregular singular point

Finally we consider expansion about an irregular (essential) singular point. In fact there is no general theory for this. Suffice to say that the solution usually cannot be written in Frobenius form.

Later we shall consider a general procedure that allows approximations to be developed for this case, for the moment note some examples

Example:

$$y''(x) + 2\frac{y'(x)}{x} - n^2\frac{y}{x^4} = 0$$

Has an essential singularity at $x = 0$. Now the general solution to this equation (see tutorial) is simply

$$y = A\exp(-n/x) + B\exp(+n/x)$$

The two independent solutions $y = \exp(\pm n/x)$ clearly cannot be written as Frobenius series about $x = 0$ since they have an essential singularity there!

Example: Airy's equation $\quad y''(x) - xy(x) = 0$

has an irregular singularity at $x = \infty$ (you should check this). As we shall see later the leading behaviour for large x is

$$y(x) \sim x^{-1/4}\exp\left\{\pm\frac{2}{3}x^{3/2}\right\}\left(1 + O(x^{-3/2})\right)$$

Again we have an essential singularity, at $x = \infty$.

Section 8 cont: *8.4 Asymptotic expansions around irregular point*

In section 8.3 we saw that the Frobenius method usually fails for expansion about an irregular singular point. Leter we develop an approximation scheme for such situations known as 'WKB approach' (Wentzel-Kramer-Brillouin, sometimes J for Jeffreys aswell) which is particularly relevant to quantum mechanical problems. Actually this method develops an asymptotic expansion. It is first useful to see how sucah an expansion arises in a simple case.

First we put our second order linear differential equation into normal form i.e. let $y(x) = v(x)u(x)$ with $v = \exp -1/2 \int^x p(x)$ then

$$u''(x) + Q(x)u(x) = 0 \tag{8}$$

$$\text{where} \qquad Q(x) = \left[q(x) - \frac{p'(x)}{2} - \frac{(p'(x))^2}{4} \right]$$

Already we can learn quite a lot about the behaviour.

Example: Bessel's equation

$$y''(x) + \frac{1}{x} y'(x) + \left(1 - \frac{n^2}{x^2}\right) y(x) = 0$$

has an essential singularity at infinity.

The normal form is given by setting $y = x^{-1/2} u$ then

$$u'' + \left[1 - \left(\frac{n^2}{x^2} - \frac{1}{4x^2} \right) \right] u = 0 \, .$$

Now for $|x| \gg 1$ this gives approximately $u'' + u \simeq 0$ thus $u \simeq A \cos x + B \sin x$ and

$$y \sim x^{-1/2} (A \cos x + B \sin x) \quad \text{as} \quad |x| \to \infty \, .$$

Note that $\sin x$ and $\cos x$ have essential singularities at ∞. For an expansion about $x = \infty$ we should replace the constant A by $\sum_{i=0}^{\infty} A_i \left(\frac{1}{x}\right)^i$ and the constant B similarly. Substsituting this expressions back into the normal form equation will generate an *asymptotic* (x large) series for the solution

Let us think generally about what (8) means by considering $Q(x)$ as if it were a constant

$Q(x) = 0$	$\Rightarrow$	point of inflection
$Q(x) > 0$	then	we expect *oscillatory* behaviour (SHM)
$Q(x) < 0$	then	we expect an *exponentially* growing or decreasing solution. Actually we generically expect an exponentially growing behaviour as $x \to \pm\infty$ and the exponentially decreasing solution is a special case.

We illustrate these ideas by a sketch in figure 1

*8. 5. *WKB approximation*

Inspired by i) the oscillating nature of our sketch for $Q(x) > 0$ and ii) the fact that for an irregular singular point of the differential equation at infinity we expect an essential

Figure 1: Sketch of solution of Schrodinger type equation (8) for $Q(x)$ as illustrated.

singularity we make the guess

$$u(x) = \exp i\phi(x) \tag{9}$$

Thus $\quad u'(x) = i\phi'(x)u(x) \qquad u''(x) = i\phi''(x)u(x) - (\phi'(x))^2 u(x)$ and we obtain

$$i\phi''(x) - (\phi'(x))^2 + Q(x) = 0\ . \tag{10}$$

To make progress with this nonlinear ODE we assume ϕ'' is 'small' so that we may ignore it. Then

$$\phi'(x) = \pm\sqrt{Q(x)} \qquad \phi(x) = \pm\int^x \sqrt{Q(t)}dt \tag{11}$$

Now (11) implies $\phi''(x) = \pm\frac{Q'}{2\sqrt{Q}}$ thus for consistency with the assumption that ϕ'' is 'small' we require

$$\left|\frac{Q'}{\sqrt{Q}}\right| \ll |Q| \tag{12}$$

We return to the interpretation of this condition presently.

To compute the next correction to (11) we treat ϕ'', as determined above, as a perturbation

$$\begin{aligned}
(\phi')^2 &= Q + i\phi'' \simeq Q \pm i\frac{Q'}{2\sqrt{Q}} \\
\text{thus} \qquad \phi' &\simeq \pm\sqrt{Q}\left(1 \mp \frac{i}{2}\frac{Q'}{Q^{3/2}}\right)^{1/2} \simeq \pm\sqrt{Q} + \frac{i}{4}\frac{Q'}{Q} \\
\text{and} \qquad \phi &\simeq \pm\int^x \sqrt{Q(t)}dt + \frac{i}{4}\ln Q
\end{aligned}$$

Inserting this final expression back into (9) yields a pair of approximate solutions

$$\boxed{u(x) \sim c_1 Q^{-1/4}(x) \exp\left[i\int^x \sqrt{Q(t)}dt\right] + c_2 Q^{-1/4}(x)\exp\left[-i\int^x \sqrt{Q(t)}dt\right]} \tag{13}$$

As it stands the WKB formula (13) is an asymptotic relation (for large x). Actually one may iterate the procedure (see tutorial) to obtain an asymptotic series of further approximations.

Example: quantum harmonic oscillator: $y'' - (x^2 - 2n - 1)y = 0$

Here the equation is already in normal form and the WKB approximations are

$$\begin{aligned} y &\sim C(x^2 - 2n - 1)^{-1/4} \exp\left[\pm \int^x (t^2 - 2n - 1)^{1/2} dt\right] \\ &\simeq Cx^{-1/2} \exp\left[\int^x \left(\pm t \mp \frac{(2n+1)}{2t} \dots\right) dt\right] \\ &= Cx^{-1/2} \exp\left[\pm \frac{x^2}{2}\right] x^{\mp(n+1/2)} \end{aligned}$$

Taking the 'physical' solution which $\to 0$ as $x \to \infty$ gives

$$y(x) \sim C \exp(-x^2/2) x^n$$

This gives the asymptotic behaviour of the quantum harmonic oscillator wavefunctions for $x \to \infty$.

*8. 6. *Schrödinger's Equation*

The WKB approximation is commonly used to determine the asymptotic behaviour of the time independent Schrödinger's Equation (S.E.) where $y(x)$ is the wavefunction

$$y''(x) + \frac{2m}{\hbar^2}(E - V(x))y(x) = 0 \tag{14}$$

Recall that for a free particle $V =$ constant and there are 'plane wave' solutions $y = e^{ikx}$ where the (constant) wavenumber $k = \sqrt{Q} = \sqrt{2m(E - V)/\hbar^2}$.

We may generalise this to the x-dependent case to think of $k(x) = \sqrt{Q(x)}$ as an 'effective' (i.e. x dependent) wavenumber. Then the effective wavelength is $\lambda(x) = 2\pi/\sqrt{Q(x)}$ and condition (12) corresponds to

$$\left|\frac{d\lambda}{dx}\right| \ll 1$$

i.e the effective wavelength is slowly varying.

The condition breaks down when $Q(x)$ vanishes or changes sharply.

The *turning points* of Q(x) are points where $Q(x) = 2m/\hbar^2(E - V(x))$ vanishes. The reason for the name becomes clear when one considers a 'classical orbit' which is restricted to $V(x) \leq E$ so that the particle has kinetic energy ≥ 0. At the turning point the kinetic energy is zero and the orbit must turn around. In the context of S.E. we expect the wavefunction $y(x)$ to decrease exponentially when $V(x) > E$ $(Q(x) < 0)$.

If $Q(x)$ vanishes at x_0, say, then (13) clearly diverges. But if $Q(x)$ is analytic at x_0 then our theory of ODEs tells us that the true solution should be regular. Thus we have a problem near the turning points.

*8. 7. *Connection problem (beyond basic scope of course)*

Let us consider a problem with one turning point i.e. one zero of $Q(x)$ which we choose for simplicty to be at $x_0 = 0$. For $x > 0$ we have $Q(x) < 0$ and for $x < 0$ we have $Q(x) > 0$.

For a physical solution ($y \to 0$ as $x \to \infty$) we require $c_2 = 0$ in (13). The question is what should the constants c_1, c_2 be for $x \ll 0$ be to correspond to the *same* solution?

The subtlety is as follows: for $x \gg 0$ and $x \ll 0$ we can use the WKB approximations (which each involve two constants). However for $|x| \ll 1$ this approximation is expected to breakdown so we don't know how to 'connect' the two limits.

What we do is to use a different approximation near the turning point that connects $x > 0$ and $x < 0$. Since we are near a zero of $Q(x)$ we can approximate $Q(x)$ by a linear function

$$Q(x) \simeq xQ'(0)$$

Then near $x = 0$, $y(x)$ satisfies

$$y''(x) - x|Q'(0)|y(x) = 0$$

which on changing variable to $z = x\,|Q'(x_0)|^{-1/3}$ is Airy's equation

$$y''(z) = zy(z)$$

The solution of this equation (that tends to zero as $z \to \infty$ is given by the 'Airy integral'

$$\mathrm{Ai}(z) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \exp\left[i\left(tz + \frac{t^3}{3}\right)\right] \tag{15}$$

(There is also a second solution that diverges as $x \to \infty$.) The asymptotics of Ai are

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} \exp(-2z^{3/2}/3) \qquad \text{for} \quad z \to +\infty \tag{16}$$

$$\mathrm{Ai}(z) \sim \frac{1}{\sqrt{\pi}} |z|^{-1/4} \cos\left(2|z|^{3/2}/3 - \pi/4\right) \qquad \text{for} \quad z \to -\infty \tag{17}$$

Note that (17) recovers the asymptotic expression we obtained by the method of stationary phase in section 4; (16) can be obtained by the saddle point method from the integral rep but it is a little subtle, instead we can deduce it from (13).

So now we can consider a point

$$x_1 \gg 0$$

at which to 'match' (i.e. equate by fixing constants) our WKB approximation to the Airy approximation for $x \gg 0$. Then we can consider a point $x_2 \ll 0$ where we match our Airy approximation (whose constant a below is now fixed) to the WKB approximation for $x \ll 0$ i.e. we construct a physical solution ($y \to 0$ for $x \gg 1$)

$$y(x) \sim c_1|Q(x)|^{-1/4} \exp\left[-\int^x |Q(t)|^{1/2} dt\right] \qquad x > x_1 \gg 0$$

$$y(x) \simeq a\mathrm{Ai}(z) \qquad \text{where} \quad z = x\,|Q'(x_0)|^{-1/3} \qquad x_2 < x < x_1$$

$$y(x) \sim C_1 Q^{-1/4}(x) \exp\left[i\int^x \sqrt{Q(t)}dt\right] + C_2 Q^{-1/4}(x) \exp\left[-i\int^x \sqrt{Q(t)}dt\right] \qquad x < x_2 \ll 0$$

Using this one can determine the precise coefficients C_1 C_2 required to correspond to the physical solution.