Section 9: Some properties of 'Special Functions'

The 'special functions' of mathematical physics are usually introduced as the solutions of certain frequently occurring second order differential equations. This is the way we encountered the Legendre polynomials and Bessel functions last section when we computed series expansions. Actually these functions have many representations:

1. Specified solutions of differential equations
2. Series
3. "Rodrigues' formula" (see below)
4. Integral representations
5. Generating functions
6. Recurrence relations

and we can use any one as a starting point for the study of the functions. In this section we shall give a flavour of how the different interrelations work for Legendre polynomials and Bessel functions. In particular we stress the utility of a generating function.

9. 1. Legendre polynomials

Recall Legendre's equation

$$(1-x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0 \ . \tag{1}$$

A solution for n integer can be found as in section 7 which we can write in a uniform way as

$$P_n(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r} \tag{2}$$

Here n is a +ve integer (n -ve integer doesn't give any new polynomials). $[n/2]$ indicates the integer part of $n/2$.

First we note that (2) can be developed as follows

$$\begin{aligned} P_n(x) &= \sum_{r=0}^{[n/2]} \frac{(-1)^r}{2^n r!(n-r)!} \frac{d^n}{dx^n} x^{2n-2r} \\ &= \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{r=0}^{n} \frac{n!}{r!\,(n-r)!} (-1)^r (x^2)^{n-r} \\ &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n \end{aligned} \tag{3}$$

(Why can we extend the limit of the sum in the second line?) The last equality gives *Rodrigues' formula.* You should write out the first few Legendre polynomials using this.

Recall from section 2 that Cauchy's integral formula gives the identity

$$\frac{d^n f(z)}{dx^n} = \frac{n!}{2\pi i}\oint \frac{f(t)}{(t-z)^{n+1}}\ dt$$

valid for f regular within the closed contour which encircles z. Thus we deduce from Rodrigues' formula that

$$P_n(x) = \frac{1}{2^n}\frac{1}{2\pi i}\oint \frac{(t^2-1)^n}{(t-x)^{n+1}} dt \tag{4}$$

which is Schläfli's integral representation that you met in tutorial 3 and from it one can conveniently extract the large n behaviour of P_n.

Now we introduce a *generating function.* This object is extremely useful for solving recurrence relations and other difference equations. Let

$$F(h,x) = \sum_{n=0}^{\infty} h^n P_n(x)$$

Here we already know P_n and we can calculate the generating function using the integral representation

$$\begin{aligned}
F(h,x) &= \frac{1}{2\pi i}\oint \frac{1}{t-x}\sum_{n=0}^{\infty}\left(\frac{h(t^2-1)}{2(t-x)}\right)^n dt \\
&= \frac{1}{2\pi i}\oint \frac{1}{t-x}\left[1-\frac{h(t^2-1)}{2(t-x)}\right]^{-1} dt \\
&= \frac{1}{2\pi i}\oint \frac{2}{[-ht^2+2t+h-2x]}\, dt \\
&= \frac{-1}{\pi i h}\oint \frac{dt}{(t-t_+)(t-t_-)}
\end{aligned}$$

where $t_\pm$ satisfy $\quad t^2 - 2t/h + 2x/h - 1 = 0 \quad \Rightarrow \quad t_\pm = \frac{1}{h}\left[1 \pm \sqrt{1-2xh+h^2}\right]$

Now as $h \to 0$ we see $t_+ \simeq 2/h \to \infty$, $t_- \to x$ thus only t_- is within the contour (small circle encircling x) and by the residue theorem

$$F(h,x) = -\frac{2}{h}\,\frac{1}{t_- - t_+} = \frac{1}{\sqrt{1-2xh+h^2}} \tag{5}$$

Let us now derive a *recurrence relation* from the generating function. We differentiate

$$\frac{\partial F}{\partial h} = \frac{x-h}{1-2hx+h^2}F \qquad \Rightarrow \qquad (1-2hx+h^2)\frac{\partial F}{\partial h} = (x-h)F$$

Equating coefficients of h^n on both sides and noting that

$$\frac{\partial F}{\partial h} = \sum_{n=0} n h^{n-1} P_n = \sum_{n=0} (n+1)h^n P_{n+1}$$

gives

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \tag{6}$$

Other recurrence relations can be derived, but (6) is sufficient to determine all Legendre polynomials from P_0.

A familiar use of the generating function is to identify the expansion useful in electrostatics

$$\begin{aligned}\frac{1}{|\underline{r}_1 - \underline{r}_2|} &= (r_1^2 - 2\underline{r}_1 \cdot \underline{r}_2 + r_2^2)^{-1/2} = \frac{1}{r_1}\left[1 - 2\frac{r_2}{r_1}\cos\theta + \left(\frac{r_2}{r_1}\right)^2\right]^{-1/2} \qquad \text{for} \quad r_1 > r_2 \\ &= \frac{1}{r_1}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_2}{r_1}\right)^n\end{aligned}$$

Numbered equations above furnish six different, but as mentioned in the preamble equivalent, representations of the Legendre Polynomials.

There remains one very important property which is *orthogonality*. The Polynomials obey

$$\int_{-1}^{+1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{nm}$$

thus they are orthogonal over the interval $[-1, +1]$. This is most easily proven using Rodrigues formula and integrating by parts (see e.g. *RHB* or *Arfken*) but we do not do so here due to lack of time. We shall return to Orthogonality of solutions of 2nd order ODEs when we consider the Sturm-Liouville problem.

9. 2. Bessel functions

We have met Bessel's equation in section 7 $\qquad x^2y'' + xy' + (x^2 - n^2)y = 0$

and we derived for n integer the Bessel functions (of the first kind)

$$J_n(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,(p+n)!}\left(\frac{x}{2}\right)^{n+2p} . \tag{7}$$

With this definition $J_{-n} = (-1)^n J_n(x)$.

Recursion relations can be deduced directly from (7) by playing around with the factorials (fiddly, so let's not do it here). One can show

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x) \tag{8}$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J'n(x) \tag{9}$$

Let us proceed to solve these equations as though we did not know formula (7). To do so we define a generating function, this time as the sum over all n from $-\infty$ to ∞

$$F(x,h) = \sum_{n=-\infty}^{\infty} h^n J_n(x) .$$

We multiply the recursion (8) by h^n and sum over all n to find

$$\left(h + \frac{1}{h}\right)F(x,h) = \frac{2h}{x}\frac{\partial F(x,h)}{\partial h}$$

This can be integrated wrt h to give

$$F = \phi(x) \exp\left[\frac{x}{2}\left(h - \frac{1}{h}\right)\right]$$

To fix $\phi(x)$ we use the second recursion (9) which implies

$$\left(h - \frac{1}{h}\right) F(h,x) = 2\frac{\partial F(h,x)}{\partial x} \qquad \Rightarrow \qquad \phi(x) = \text{constant}$$

Taking the constant as one yields the usual normalisation of J_n. We are now left with the problem of inverting $F(x,h)$ to give $J_n(x)$. This is generally the most difficult step in solving a recurrence relation. One approach is just to expand the generating function in powers of h using brute force. However a more elegant approach is to note that since $F(h,x)$ is a series in powers of h we may use the residue theorem to give

$$J_n(x) = \frac{1}{2\pi i}\oint \frac{F(t,x)}{t^{n+1}}\, dt = \frac{1}{2\pi i}\oint \frac{\exp\left[\frac{x}{2}\left(t - t^{-1}\right)\right]}{t^{n+1}}\, dt \tag{10}$$

where the contour encircles the origin. This gives an integral representation of the Bessel function. Another integral rep may be obtained by letting $t = e^{i\phi}$ then $dt = itd\phi$ and

$$\begin{aligned} J_n(x) &= \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\left[\frac{x}{2}\left(e^{i\phi} - e^{-i\phi}\right) - in\phi\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp\left(ix\sin\phi - in\phi\right) d\phi = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi - n\phi) d\phi \end{aligned}$$

9. 3. Generalisation of Bessel functions to non integer order

If $n = \nu$ is not an integer we can continue the series representation (7) by using the Gamma function:

$$J_\nu(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,\Gamma(p+\nu+1)!}\left(\frac{x}{2}\right)^{\nu+2p} \tag{11}$$

and $J_{-\nu}$ and J_ν are now independent functions

However to generalise the integral representation we have to be careful since $t^{\nu+1}$ in (10) implies a branch cut. We take the branch cut from 0 to $-\infty$ along the negative real axis then one can show that

$$J_\nu(x) = \frac{1}{2\pi i}\int_C \frac{\exp\left[\frac{x}{2}\left(t - t^{-1}\right)\right]}{t^{\nu+1}}\, dt \tag{12}$$

where the contour C begins just below the branch cut at $-\infty - i\epsilon$ goes around the origin then goes off to $-\infty + i\epsilon$ (just above the branch cut). The idea of this 'Hankel' or 'loop' type contour is that the integrand vanishes at the end-points.

If one writes the contour as a sum of two pieces from $-\infty - i\epsilon$ to the origin then from the origin to $-\infty + i\epsilon$ one has

$$J_\nu(x) = H_\nu^{(2)}(x)/2 + H_\nu^{(1)}(x)/2$$

where $H_\nu^{(2)}(x)$,$H_\nu^{(1)}(x)$ are Hankel functions of the second and first kind which we met in section 5 and which are also linearly independent solutions of Bessel's equation.