Section 13: Partial Differential Equations

Initially we will be considering PDEs in two dimensions thus the independent variables are x, y or x, t if we think of one dimension being time. The dependent variable (i.e. the function that satisfies the PDE) will usually be denoted u and sometimes by ψ (e.g. in quantum mechanics).

A first order PDE is a relation of the form (where we use the shorthand $u_x = \dfrac{\partial u}{\partial x}$ etc)

$$F(x, y, u, u_x, u_y) = 0 \ .$$

The equation is *linear* if F is a linear function of u, u_x, u_y.

Similarly a second order PDE is a relation of the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

and so on.

In general the solution of a PDE will involve arbitrary *functions*. This is a key difference from say an n^{th} order ODE where we have n arbitrary *constants* of integration. We will show how to specify these arbitrary functions through the imposition of boundary conditions.

Simple example

Consider the first order linear PDE

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 \qquad a,\ b \quad \text{constant} \tag{1}$$

Let us think about this equation geometrically. Recall that the directional derivative of a function u in a direction $\underline{\hat{n}}$ is

$$\frac{\partial u}{\partial n} = \underline{\hat{n}} \cdot \underline{\nabla} u \ .$$

Thus the PDE (1) reads that the derivative in the direction parallel to (a, b) is zero i.e. u is constant along straight lines

$$bx - ay = \quad \text{constant} \ .$$

Such curves are known as 'characteristics':

The general solution is then $u = f(bx - ay)$ where $f(z)$ is an arbitrary function of one variable.

Now sketch the characteristics. Consider specifying $u(x, 0)$ as the boundary condition. Since u is constant along each characteristic we in effect specify the solution everywhere in the x–y plane. Similarly we could specify the solution everywhere by the boundary condition $u(0, y)$. Generally an appropriate boundary condition is to specify u along an *open* curve i.e. one that a given characteristic does not cross more than once.

Generally **Characteristics** are

- curves along which partial information about the solution propagates from the boundary curve.
- curves on which one cannot impose arbitrary boundary conditions

Figure 1: Characteristics for equation 1

Aside: For a general class of first order PDE of the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

(sometimes known as 'quasilinear') the general solution can be obtained by a procedure known as the 'Method of Characteristics'.

13. 1. Second order PDEs

Examples from physics:

	Equation	2d version
Wave	$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$	$u_{xx} - \frac{1}{c^2}u_{tt} = 0$
Laplace's	$\nabla^2 u = 0$	$u_{xx} + u_{yy} = 0$
Diffusion	$\nabla^2 u - \frac{1}{\kappa}\frac{\partial u}{\partial t} = 0$	$u_{xx} - \frac{1}{\kappa}u_t = 0$
S.E. (time dependent)	$-i\hbar\frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi + V\psi = 0$	$i\hbar\psi_t + \frac{\hbar^2}{2m}\psi_{xx} + V\psi = 0$
S.E. (time independent)	$\nabla^2\psi + \frac{2m}{\hbar^2}[E-V]\psi = 0$	$\psi_{xx} + \psi_{yy} + \frac{2m}{\hbar^2}[E-V]\psi = 0$

All these equations are *homogeneous.* Inhomogeneous versions are also of interest e.g.

$$\text{Poisson's equation} \qquad \nabla^2 u = f(\underline{r}, t)$$

We will consider the general class of second order PDE of the form

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = D(x,y,u_x,u_y) \tag{2}$$

We now determine the **characteristics** by trying to impose 'Cauchy boundary conditions' along a curve Γ parametrised as $x = X(s)$ and $y = Y(s)$ where s is the arclength. Cauchy boundary conditions comprise fixing u and $\frac{\partial u}{\partial n}$ (the normal derivative) along Γ. This amounts to specifying u and u_x, u_y. To see this recall that the tangent vector to the curve and the normal vector to the curve are given by

$$\hat{\underline{t}} = (X', Y') \qquad \hat{\underline{n}} = (Y', -X')$$

Then

$$\frac{\partial u}{\partial n} = \hat{\underline{n}} \cdot \underline{\nabla} u = Y' u_x - X' u_y \quad \text{and} \quad \frac{\partial u}{\partial s} = \hat{\underline{t}} \cdot \underline{\nabla} u = X' u_x + Y' u_y$$

Since u is specified along Γ so is $\partial u/\partial s$ thus the above two equations yield u_x and u_y. Thus Cauchy b.c. also specify $\partial u_x/\partial s$ and $\partial u_y/\partial s$ along Γ. Now consider

$$\begin{aligned} \frac{\partial u_x}{\partial s} &= \hat{\underline{t}} \cdot \underline{\nabla} u_x = X' u_{xx} + Y' u_{xy} \\ \frac{\partial u_y}{\partial s} &= \hat{\underline{t}} \cdot \underline{\nabla} u_y = X' u_{xy} + Y' u_{yy} \end{aligned}$$

These along with (2) give three equations for three unknowns u_{xx}, u_{xy}, u_{yy}. The question is are these consistent i.e. is there a solution? We write the equations in matrix form

$$\begin{pmatrix} A & 2B & C \\ X' & Y' & 0 \\ 0 & X' & Y' \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} D \\ \partial u_x/\partial s \\ \partial u_y/\partial s \end{pmatrix}$$

All the components of the rhs are specified by the boundary data. Thus for a solution we require that

$$\begin{vmatrix} A & 2B & C \\ X' & Y' & 0 \\ 0 & X' & Y' \end{vmatrix} \neq 0$$

The characteristics are deduced from when this condition does *not* hold i.e. there is no solution for the second derivatives thus Cauchy boundary conditions along Γ are inconsistent with the PDE:

$$A Y'^2 - 2B X' Y' + C X'^2 = 0$$

$$\text{divide by} \quad X'^2 \qquad A \left(\frac{dY}{dX}\right)^2 - 2B \left(\frac{dY}{dX}\right) + C = 0$$

$$\Rightarrow \qquad \frac{dY}{dX} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

this gives the tangent to the characteristics at any point. If A, B, C are constants we get straight lines.

We classify a PDE (at a given point) according to how many real characteristics there are

	classification	**characteristics**
$B^2 - AC < 0$	elliptic	two complex
$B^2 - AC > 0$	hyperbolic	two real
$B^2 - AC = 0$	parabolic	one real

N.B. an equation can be of a different type at different points if A, B, C are functions of x, y.

13. 2. Normal forms

Idea: transform to 'normal co-ordinates' i.e. choose new independent variables $v(x,y), w(x,y)$ so that the coefficient of $\partial^2/\partial v \partial w$ vanishes in (2)

For simplicity consider A, B, C constant so that the transformation is the same at all points. In this case the transformation is linear and can be expressed in terms of the characteristics which are given by

$$\alpha_{\pm}(x,y) = y - \frac{B \pm \sqrt{B^2 - AC}}{A} x = \text{constant}$$

Here let us just summarise the results of the transformation.

equation type	**change of variables**	**new equation**	
elliptic	$v = \frac{1}{2}(\alpha_+ + \alpha_-) \quad w = -\frac{i}{2}(\alpha_+ - \alpha_-)$	$\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial w^2} = \widetilde{D}$	'Laplace-like'
hyperbolic	$v = \frac{1}{2}(\alpha_+ + \alpha_-) \quad w = \frac{1}{2}(\alpha_+ - \alpha_-)$	$\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial w^2} = \widetilde{D}$	'wave-like'
parabolic	$v = \alpha \quad w = x$	$\frac{\partial^2 u}{\partial w^2} = \widetilde{D}$	'diffusion-like'

Thus Laplace's, the Wave and the Diffusion equation are canonical examples i.e. typical of each class and we shall study them in the next few sections.

13. 3. Boundary conditions

First let us consider the simplest hyperbolic equation: the wave equation

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0$$

the characteristics are given by $\left(\frac{dt}{dx}\right)^2 = \frac{1}{c^2}$ i.e. $\alpha_+ = x + ct = \text{constant}$ and $\alpha_- = x - ct = \text{constant}$: α_+ is backwards moving; α_- is forwards moving The general solution is

$$u = f(x - ct) + g(x + ct)\ .$$

Thus if we specify u and u_x, u_y along some open boundary curve e.g. a portion of the x-axis this fixes f and g along the curve and in a quadrilateral delineated by the characteristics starting from the rightmost and left most ends of the boundary curve. Hence the solution of the equation is determined within this quadrilateral. (You should sketch this in a figure).

Thus for a **hyperbolic** equation Cauchy boundary conditions along some open curve (not a characteristic) are the appropriate b.c.s.

For a **parabolic** equation we have only one family of characteristics and therefore only one arbitrary function to fix. Therefore we expect that we should specify boundary conditions along an open curve but we should only specify **either** u on the curve (Dirichlet b.c.s) **or** $\partial u/\partial n$ (Neumann b.c.s)

For an **elliptic** equation there are no real characteristics therefore we specify boundary conditions on a closed surface (somehow the information 'seeps in' from the boundary conditions) and use either Dirchlet or Neumann conditions.