METHODS OF MATHEMATICAL PHYSICS

Infinite series; Residues; Analytic Continuation **Tutorial Sheet 1**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

1.1 ‘Telescoping’ an infinite series [S]

Consider $S = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}$

By writing $n = (n+1) - 1$ in the numerator of the summand, show that a large cancellation occurs between the resulting two series. Hence evaluate S.

1.2 Comparison of a series with an integral [S]

By comparison with an integral, show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^s}$$

converges for $s > 1$.

What does the ratio test yield for this series?

1.3 Bernoulli Numbers [C]

Consider the infinite series

$$\frac{x}{e^x - 1} = B_0 + B_1 x + \frac{B_2}{2!}x^2 \cdots + \frac{B_n}{n!}x^n \cdots$$

By expanding the lhs and equating powers of x determine that

$$1 = B_0$$
$$0 = \frac{B_0}{2!} + \frac{B_1}{1!}$$
$$0 = \frac{B_0}{3!} + \frac{B_1}{2!\ 1!} + \frac{B_2}{1!\ 2!}$$

Show that for $n > 1$ one can write the relations as

$$(B+1)^n - B^n = 0 \qquad \text{where} \quad B^s \to B_s$$

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1.4 Asymptotic expansion of error function [K]

The error function is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\ e^{-t^2}$$

Show by considering the complementary error function $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ and integrating by parts that

$$\mathrm{erf}(x) \sim 1 - \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} \sum_{n=0}^{\infty} \frac{(-1)^n\ (2n-1)!!}{x^{2n} 2^n}$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots(3)(1)$

Explain why this is an asymptotic expansion of the error function.

1.5 Analytic Continuation [K]

Suppose we are given the power series expansion

$$f(z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$$

which converges for $|z| < 1$.

Sum this series and derive a power series expansion for the resulting function about $z = -1/2$. What is the radius of convergence of this series?

Repeat for expansions about $z = 1/2$ and $z = 3/4$.

Discuss the complication that arises when analytically continuing to the point $z = 2$.

1.6 Bernoulli numbers from a contour integration [S]

(i) Show that generalising x to complex variable z (analytic continuation) in question 1.3 implies that

$$B_n = \frac{n!}{2\pi i} \oint_C \frac{z}{e^z - 1} \frac{dz}{z^{n+1}}$$

where the contour C encircles the origin in an anticlockwise fashion with $|z| < 2\pi$ to avoid the poles at $\pm 2\pi i$.

(ii) Locate and classify all the singularities of the integrand.

(iii) By the residue theorem $B_n = 2\pi i \times$ residue from origin. By deforming the contour show that

$B_n = -2\pi i \times$ sum of all other residues

Hint: you will need to consider a large circle of radius $R \to \infty$ encircling the origin in a *clockwise* fashion

(iv) Hence show that

$$\begin{aligned} B_n &= 0 \quad \text{for} \quad n \text{ odd} \\ B_n &= -2\frac{(-1)^{n/2} n!}{(2\pi)^n}\zeta(n) \quad \text{for} \quad n \text{ even} \end{aligned}$$

where $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$. By calculating explicitly the Bernouilli numbers in 1.2, evaluate $\zeta(2)$ and $\zeta(4)$

1.7 Analytic continuation in the context of an integral

Consider the integral

$$I = \int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x + 1} dx \qquad (0 < a < 1)$$

(i) Evaluate the integral by closing the contour in the upper half plane and using residues. Note that to ensure that the large semi-circle does not give a contribution to the integral you must introduce a small imaginary part to a. Sum the resulting residues

(ii) Why can we then take the limit of a real to deduce the orginal integral?

Ans: $I = \dfrac{\pi}{\sin \pi a}$

(iii) Why can't we analytically continue to $a > 1$?