METHODS OF MATHEMATICAL PHYSICS

Gamma Function; Laplace's Method **Tutorial Sheet 2**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

2.1 **Generalising the gaussian integral formula** [s] Given the formula

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}$$

show that:

(i)

$$\int_{-\infty}^{\infty} dx\, e^{-ikx-ax^2/2} = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a}$$

(Hint: try completing the square, then close a contour in the complex plane)

(ii)

$$\int_{-\infty}^{\infty} dx\, e^{iax^2/2} = \sqrt{\frac{2\pi}{a}} e^{i\pi/4}$$

2.2 **Use of Gamma function** [s] Considering the integral

$$\int \mathrm{d}\,\underline{x} e^{-r^2}$$

over the n dimensional unit sphere where $\underline{x}$ is the n-dimensional position vector and r is the radial distance. By evaluating the integral in two ways— i) as a product of n one-dimensional integrals over x_i ii) as a one dimensional integral over r— express the surface area and volume of the n dimensional unit sphere in terms of Gamma functions (N.B. a circle is 2d sphere, usual sphere is 3d sphere etc)

2.3 **Another use of Gamma function** [s] Show that

$$\int_0^{\infty} e^{-s^p} ds = \frac{\Gamma(1/p)}{p}$$

2.4 **Generalising Laplace's Method** [s] Generalise Laplace's method to calculate the leading approximation to the integrals along the real axis of the form

$$I(x) = \int_a^b f(t) e^{x\phi(t)}\, dt \qquad \text{for} \quad x \gg 0$$

if the near to the stationary point the expansion of ϕ is

$$\phi(t) = \phi(c) + \frac{1}{n!}(t-c)^n\, \phi^{(n)}(c) + \cdots$$

where n is even and $\phi^{(n)}(c) < 0$. You will need to use the result of Q2.3

2.5 **Derivation of Euler's reflection formula** [r] Review the derivation of

$$\Gamma(z)\Gamma(1-z) = \Gamma(1)B(z,1-z) = \int_0^1 dt\ t^{z-1}(1-t)^{-z} = \int_0^\infty dx\ \frac{x^{z-1}}{1+x}$$

where we changed variables $t = x/(1+x)$.

Evaluate the final integration by a contour integral to show

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}\ .$$

Hint: choose the contour to be the same as Hankel's contour.

Why is the resulting expression valid for whole complex plane?

2.6 **Hypergeometric Function** [c] Consider the hypergeometric function defined as

$${}_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\ \frac{x^n}{n!}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$.

Use the Beta function to verify the integral representation

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$$

2.7 **Stirling's Formula** [r] Review the derivation of Stirling's formula by Laplace's method

$$\begin{aligned}
\Gamma(x+1) &= x^{x+1}\int_0^\infty ds\ \exp(x\,[-s+\ln s]) \\
&\simeq x^{x+1}e^{-x}\int_{-\infty}^\infty ds\ \exp(-xu^2/2) \\
&= x^{x+1/2}e^{-x}\sqrt{2\pi}
\end{aligned}$$

Now consider calculating the next term in the expansion. Derive the general formula

$$\int_{-\infty}^\infty du\, u^n\, e^{-au^2/2} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{\sqrt{2\pi}}{a^{(n+1)/2}}(n-1)(n-3)(n-5)....(3)(1) & \text{if } n \text{ even} \end{cases} \qquad (1)$$

Using this formula work out to which order you have to expand $-s + \ln s$ to calculate the first correction to Stirling's formula and identify the integrals that will contribute.

2.8 **Watson's Lemma** [c] Consider

$$I(x) = \int_0^b f(t)e^{-xt}dt \qquad b > 0\ .$$

If $f(t)$ has an asymptotic series expansion for t small

$$f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n} \quad \alpha > -1 \quad \beta > 0$$

show that

$$I(x) \sim \sum_{n=0}^\infty \frac{a_n\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}}\ .$$