METHODS OF MATHEMATICAL PHYSICS

Fourier Transformations; Solution of ODEs **Tutorial Sheet 6**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

6.1 **Fourier Transforms [k]**

What is the Fourier transform of

(i) $\delta(x)$ (ii) $\delta'(x)$ (iii) $\theta(x)$

(iv) $1/x$ (v) $\exp(-\frac{1}{2}\alpha x^2)$?

6.2 **Fourier series of delta function [s]**

(i) Construct the Fourier series, over the interval $[-L, L]$, of the 'Top hat' delta sequence

$$\delta_n(x) \equiv \begin{cases} 0 & x < -\frac{1}{n} \\ \frac{n}{2} & -\frac{1}{n} < x < \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

(ii) Take the limit $n \to \infty$ and let $L = \pi$ to obtain the following infinite sum representation of the $\delta(x)$

$$\delta(x) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{imx}$$

(iii) Obtain the Fourier sine series on the interval $[0, L]$

$$\delta(x - x') = \frac{2}{L} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x'}{L}\right) \sin\left(\frac{m\pi x}{L}\right)$$

and similarly obtain a cosine series

6.3 **Convolution Theorem [k]**

Show that

$$\mathcal{F}[f_1(x)f_2(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(k')g_2(k - k')dk'$$

where $\mathcal{F}$ denotes Fourier transform and $g_1(k) = \mathcal{F}[f_1]$, $g_2(k) = \mathcal{F}[f_2]$.

6.4 **Parseval's Theorem [s]**

For the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \exp\frac{in\pi x}{L} \,.$$

and Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, g(k)e^{ikx}$$

Show that

$$\frac{1}{2L} \int_{-L}^{L} dx\, |f(x)|^2 = \sum_{n=-\infty}^{\infty} |C_n|^2 \qquad \text{and} \qquad \int_{-\infty}^{\infty} dx\, |f(x)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, g(k)g^*(k)$$

6.5 **Fourier transform solution of forced SHM [s]**

By the method of Fourier transform find the *general* solution to the equation

$$y'' + \omega^2 y = \sin(\alpha x)$$

6.6 **Causal and non-causal systems [s]**

Use the Fourier transforms to obtain solutions to the equations

(i) $f''(t) + 2f'(t) + f(t) = g(t)$

(ii) $f'''(t) - 2f'(t) + 4f(t) = g(t)$

in the form of a convolution. Which of these equations describes a causal linear system?

Hint for (ii): one root of $ik^3 + 2ik - 4 = 0$ is $k = 2i$