METHODS OF MATHEMATICAL PHYSICS

More Green Functions; Laplace Transforms **Tutorial Sheet 7**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

7.1 **Green function for Bessel's equation** [s] Show that the Green function for the problem

$$\mathcal{L}(x)y(x) = \frac{d}{dx}\left\{x\frac{dy}{dx}\right\} + \left\{k^2x - \frac{n^2}{x}\right\}y = f(x) \qquad \text{for} \quad 0 < x < a$$

such that $y(0)$ is bounded and $y(a) = 0$, is

$$G(x, x') = \frac{\pi}{2}\left\{J_n(ka)N_n(kx') - N_n(ka)J_n(kx')\right\}\frac{J_n(kx)}{J_n(ka)}\,,$$

for $0 < x < x'$, provided $J_n(ka) \neq 0$.

Obtain the Green function for the region $x' < x < a$

[The LI independent solutions of $\mathcal{L}(x)y(x) = 0$ are $J_n(kx)$ and $N_n(kx)$, the Bessel and Neumann functions of order n. $J_n(z)$ is bounded as $z \to 0$ and $N_n(z)$ diverges as $z \to 0$ and their Wronskian is $2/\pi z$]

7.2 **Hermiticity [k]**

In order that Sturm Liouville differential operator

$$\mathcal{L}(x) = \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$$

be Hermitian i.e $\int dx u^*\mathcal{L}v = \left[\int dx v^*\mathcal{L}u\right]^*$

what are the boundary conditions that must be imposed on u, v?

7.3 **Bilinear form of Green Function [r]**

Show that the Green function for the problem

$$y''(x) + \omega^2 y(x) = f(x) \quad \text{for} \quad 0 < x < L\,; \qquad y(0) = y(L) = 0$$

is

$$G(x, x') = -\frac{\sin\omega x\sin\omega(L-x')}{\omega\sin\omega L}\theta(x'-x) - \frac{\sin\omega x'\sin\omega(L-x)}{\omega\sin\omega L}\theta(x-x')\,.$$

Find also a bilnear form for this Green function, by expanding in terms of the eigenfunctions of the above differential equation (with $f(x) = 0$) and boundary conditions.

** Verify that the two forms of the Green function are equivalent

7.4 **Symmetry of Green function** [c] Show that the Green function for an Hermitian differential operator $\mathcal{L}$ satisfies the symmetry relation

$$G(x, x') = [G(x', x)]^*$$

7.5 **[k]** (a) Given that $F(s)$ is the Laplace transform of $f(t)$, such that

$$F(s) \;=\; \mathcal{L}[f(t)] \;=\; \int_0^\infty e^{-st} f(t)dt\,,$$

verify the following basic Laplace transform properties:–

$$\text{(i)}\quad \mathcal{L}[e^{\alpha t} f(t)] \;=\; F(s-\alpha); \qquad \text{(ii)}\quad \mathcal{L}[f(t-\alpha)\theta(t-\alpha)] \;=\; e^{-\alpha s} F(s);$$

$$\text{(iii)}\quad \mathcal{L}[f(at)] \;=\; \frac{1}{a}F\Big(\frac{s}{a}\Big); \qquad \text{(iv)}\quad \mathcal{L}[t^n f(t)] \;=\; \Big(-\frac{d}{ds}\Big)^n F(s).$$

(b) Given that the Laplace transform of $\cos t$ is $s/(s^2+1)$, find using the above results the Laplace transforms of

$$\text{(i)}\quad \cos\omega(t-t_0)\theta(t-t_0); \qquad \text{(ii)}\quad e^{-\kappa t}\cos\omega t; \qquad \text{(iii)}\quad t\,\cos\omega t\,.$$

7.6 (i) Given that $\mathcal{L}[\sin\omega t] = \omega/(s^2+\omega^2)$, use the convolution theorem to find $\mathcal{L}^{-1}[(s^2+\omega^2)^{-2}]$.
(ii) Given that $\mathcal{L}[(\pi t)^{-1/2}] = s^{-1/2}$, use the convolution theorem to show that

$$\mathcal{L}^{-1}[1/s\sqrt{s+1}] = \mathrm{erf}(\sqrt{t}),$$

where $\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x du e^{-u^2}$.

7.7 Use the Bromwich inversion formula and the residue theorem to obtain the inverse Laplace transforms of the following functions:

(i) $s(s^2+a^2)^{-1}(s^2+b^2)^{-1}$, (ii) $(s^2+\omega^2)^{-2}$, (iii) $e^{-bs}/(s-a)^n$,
(iv)* $s^{-1/2}$; (v)** $(s\,\cosh s)^{-1} = 2s^{-1}\sum_0^\infty(-)^n e^{-(2n+1)s}$.

In the last part you can compare the results obtained by inverting both the given expressions to show that $\sum_0^\infty \frac{(-)^n}{2n+1} = \frac{\pi}{4}$.

7.8 If $f(t)$ is periodic, such that $f(t+T) = f(t)$, show that its Laplace transform is

$$F(s) \;=\; \Big(1-e^{-sT}\Big)^{-1}\int_0^T e^{-st} f(t)dt\,.$$

Find $F(s)$ if $f(t) = |\sin\omega t|$, the output of a full-wave rectifier.

7.9 Use the Laplace transform to solve the following initial value problems:

(i) $\ddot{x} - x = t, \qquad x(0) = \dot{x}(0) = 0,$

(ii)* $\ddot{x} + \omega^2 x + \alpha^2(x-y) = 0,$
$\ddot{y} + \omega^2 y + \alpha^2(y-x) = 0,$
$x(0) = y(0) = \dot{y}(0) = 0, \qquad \dot{x}(0) = u.$

(iii) $x''' + x'' + x' + x + 1 = 0,$
$x(0) = x'(0) = x''(0) = 0.$

(iv)** $\ddot{\underline{r}} + \underline{k}\times\underline{r} = 0,$
$\underline{r}(0) = \underline{a}, \qquad \dot{\underline{r}}(0) = 0$, where $\underline{k}$ is a constant unit vector.