METHODS OF MATHEMATICAL PHYSICS

More on Laplace Transforms; Partial Differential Equations **Tutorial Sheet 8**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

8.1 **An integral equation** [s] Use Laplace transforms to solve the integral equation

$$f(y) = 1 + \int_0^y xe^{-x}\, f(y-x)\, dx.$$

8.2 **Asymptotic behaviour** [c] In the system of first order differential equations

$$\dot{\underline{u}} = \boldsymbol{A}\underline{u} + \underline{\alpha}e^{i\omega_0 t}, \qquad \underline{u}(0) = 0,$$

for the complex n dimensional vector $\underline{u}(t)$, $\underline{\alpha}$ is a constant complex vector, $\boldsymbol{A}$ is a constant complex $n \times n$ matrix, and ω_0 is a real constant. By taking Laplace transforms show that

$$\underline{u}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (s\mathbb{1} - \boldsymbol{A})^{-1}\underline{\alpha}\frac{e^{st}}{s - i\omega_0} ds,$$

where $c > \max\{0, \Re(\lambda_j)\}$, $\{\lambda_j\}$ being the eigenvalues of $\boldsymbol{A}$. Hence show that if $\Re(\lambda_j) < 0$, $j = 1, 2, \ldots n$, then

$$\underline{u}(t) = (i\omega_0\mathbb{1} - \boldsymbol{A})^{-1}\underline{\alpha}e^{i\omega_0 t} + \underline{v}(t),$$

where $\underline{v}(t)$ is a vector which vanishes exponentially as $t \to \infty$.

8.3 **Laguerre Polynomials** [s] (i) The Laguerre polynomials may be defined as

$$L_n(t) = e^t \frac{d^n}{dt^n}(t^n e^{-t}).$$

Show that $\mathcal{L}[L_n(t)] = n!(s-1)^n/s^{n+1}$. Now consider the Laplace transform of f_n satisfying the equation

$$t\ddot{f}_n + (1-t)\dot{f}_n + nf_n = 0.$$

Deduce that $f_n = L_n$ is a solution to this equation.

(ii) By taking its Laplace transform show that $F(x,t) = e^{-xt/(1-x)}/(1-x)$ is a generating function for the Laguerre polynomials, ie that

$$F(x,t) = \sum_{n=0}^{\infty} \frac{x^n}{n!} L_n(t).$$

8.4 **Riemann Zeta function** [s] The zeta function is defined for $\Re z > 1$ by $\zeta(z) = \sum_1^\infty s^{-z}$. By considering $\int_0^\infty e^{-st}t^{z-1}dt$ show that

$$\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t - 1} dt.$$

Deduce from this the Hankel representation

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i}\int_C \frac{t^{z-1}}{e^{-t} - 1} dt,$$

where C is the anticlockwise 'loop' contour around the branch cut

from $-\infty$ to 0 (see lecture 3).

[You will need to recall Euler's reflection formula]

8.5 **Heaviside expansion theorem** [s]

If the Laplace transform $F(s)$ may be written as a ratio

$$F(s) = \frac{g(s)}{h(s)}$$

where $g(s)$ and $h(s)$ are analytic functions, $h(s)$ having simple isolated zeros at $s = s_i$ show that

$$f(t) = L^{-1}\left[\frac{g(s)}{h(s)}\right] = \sum_i \frac{g(s_i)}{h'(s_i)} e^{s_i t}$$

8.6 **Bessel's Equation of order 0** [s]

(i)

$$t\ddot{f}(t) + \dot{f}(t) + tf(t) = 0$$

Let $F(s)$ be the Laplace Transform of $f(t)$. Show that

$$-(s^2+1)\frac{dF(s)}{ds} = sF(s)$$

Integrate this equation to obtain

$$F(s) = \frac{A}{(s^2+1)^{1/2}}$$

A is the constant of integration. $A = 1$ gives the Bessel function J_0.

(ii) The inversion integral gives

$$J_0(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{ds\ e^{st}}{(s^2+1)^{1/2}}$$

Identify the singularities of $F(s)$ and discuss how to close the integration contour and whether branch cuts need be introduced.

(iii)* By using the inversion formula and expanding $F(s)$ about the singularities determine the large the asymptotic behaviour of $J_0(t)$ as $t \to \infty$ and the first correction to this behaviour.

(iv) Now develop a small t expansion by expanding $F(s)$ for large s then using the inversion formula. You should show

$$J_0(t) = \sum_{r=0}^{\infty} \frac{(-)^r(\frac{1}{2}t)^{2r}}{r!\,r!}.$$