METHODS OF MATHEMATICAL PHYSICS

Solution of Partial Differential Equations **Tutorial Sheet 9**

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

9.1 **D'Alembert's formula and characteristics** [r]

Consider the wave equation in one dimension

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0$$

(i) By changing to variables $\xi = x - ct$, $\eta = x + ct$ show that the wave equation has the form

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

and that this has the general solution

$$u = f(\xi) + g(\eta) = f(x - ct) + g(x + ct)$$

where f and g are general functions

(ii) What is the interpretation of the two contributions $f(x - ct)$ and $g(x + ct)$?

(iii) Suppose that u and u_t are given at $t = 0$:

$$u(x, 0) = U(x) \qquad u_t(x, 0) = V(x)$$

Show that these conditions determine the functions

$$\begin{aligned} f(x) &= \frac{U(x)}{2} - \frac{1}{2c} \int_a^x dx' V(x') \\ g(x) &= \frac{U(x)}{2} + \frac{1}{2c} \int_a^x dx' V(x') \end{aligned}$$

where a is an arbitrary constant. Hence obtain D'Alembert's formula

9.2 **Laplace's equation and characteristics** [k]

Use the method of 9.1 (i) to show that the general solution to Laplace's equation in two dimensions

$$u_{xx} + u_{yy} = 0$$

is

$$u(x, y) = f(x + iy) + g(x - iy)$$

with f and g arbitrary functions. If u is to be a real function show that this solution reduces to

$$u(x, y) = 2\Re f(x + iy)$$

9.3 **Inhomogeneous diffusion equation** [s]

The density $\rho(x, t)$ of a radioactive gas diffusing into the atmosphere ($x > 0$) from the ground ($x < 0$) satisfies the equation

$$\frac{\partial^2 \rho}{\partial x^2} = \frac{1}{\kappa} \frac{\partial \rho}{\partial t} + \gamma \rho \qquad (\text{for} \quad x > 0) \tag{1}$$

Assuming that $\rho(x, 0) = 0$ and $\rho_x = -\alpha$ at $x = 0$ and that $\rho(x, t) \to 0$ as $x \to \infty$:

(i) Use a Fourier cosine transform to obtain the solution

$$\rho(x,t) = \frac{2\alpha}{\pi}\int_0^\infty \frac{1-\exp[-\kappa(\gamma+y^2)t]}{\gamma+y^2}\cos(xy)dy$$

(ii) Use a Laplace transform with respect to t to obtain a solution to (1) in the form

$$\rho(x,t) = \int_0^t g(x,\tau)d\tau$$

giving an expression for $g(x,\tau)$

(iii) Demonstrate that the two solutions are equivalent by showing that (i) satisfies

$$\frac{\partial\rho}{\partial t} = g(x,t)$$

9.4* Laplace transform of erfc [c]

(i)* if $L[f(x,t)] = \dfrac{\exp(-a\sqrt{s})}{s}$ show by using the inversion integral and deforming it into a loop integral around a branch cut, that

$$f(x,t) = \mathrm{erfc}(a/2t^{1/2}) \qquad \text{where} \quad \mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}}\int_x^\infty due^{-u^2}\ .$$

(ii) Conversely, if $f(t) = \mathrm{erfc}(a/2t^{1/2})$, show that $g(t) \equiv f'(t)$ is

$$g(t) = a\exp\left\{-a^2/4t\right\}/2\sqrt{\pi t^3}\,.$$

**Show that $G(s) \stackrel{\text{def}}{=} L[g(t)] = \exp(-as^{1/2})$, and deduce that $L[f(t)] = \exp(-as^{1/2})/s$. [Hint: calculate $G'(s)$ and thereby show that $G(s)$ satisfies the (simple) differential equation $G'(s) = -a\,G(s)/2s^{1/2}$. A boundary condition for this equation can be found by evaluating $G(0)$ explicitly.]

9.5 Heat conduction problem [s]

The temperature $u(x,t)$ within a uniform, insulated bar satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t} \qquad (\text{for } -l < x < l)$$

At time $t=0$ the bar is at a constant temperature Θ and for $t>0$ the two ends ($x=\pm l$) are kept at zero temperature.

(i) Using a Fourier cosine series show that

$$u(x,t) = \frac{4\Theta}{\pi}\sum_{n\ \mathrm{odd}}\frac{1}{n}\exp\left\{-\frac{n^2\pi^2\kappa t}{4l^2}\right\}\cos\left\{\frac{n\pi(x-l)}{2l}\right\}$$

(ii) Taking the Laplace transform of the original equation, where $F(x,s)$ is the Laplace transform of $u(x,t)$, show that

$$F(x,s) = \frac{\Theta}{s}\left\{1-\frac{\cosh(x\sqrt{s/\kappa})}{\cosh(l\sqrt{s/\kappa})}\right\}\ .$$

By using $\text{sech}\alpha = 2\sum_{n=0}^{\infty}(-1)^n \exp -(2n+1)\alpha$, show that the solution can be expressed as

$$u(x,t) = \Theta - \Theta\sum_{n=0}^{\infty}(-1)^n \text{erfc}\left\{\frac{(2n+1)l - x}{2\sqrt{\kappa t}}\right\} - \Theta\sum_{n=0}^{\infty}(-1)^n \text{erfc}\left\{\frac{(2n+1)l + x}{2\sqrt{\kappa t}}\right\}$$

(iii) Discuss in which regimes are the two expansions found above most useful.

9.6 **Green function for Helmholtz operator in 1d** [s]

Show that the Green function for the modified Helmholtz operator in one dimension, $\mathcal{L}(x) = \frac{d^2}{dx^2} - k^2$, subject to the boundary condition that it should vanish at $x \to \pm\infty$ is

$$G(x, x') = -\frac{1}{2k} e^{-k|x-x'|}$$

9.7 **Fundamental Solution of Diffusion equation** [r]

Use the Fourier transform to calculate the fundamental Green function for the diffusion equation in 3d

$$\nabla^2 G(\underline{x} - \underline{x}', t - t') - \frac{1}{D}\frac{\partial G(\underline{x} - \underline{x}', t - t')}{\partial t} = \delta(\underline{x} - \underline{x}')\delta(t - t')$$

9.8 **Green function for Laplace's equation in 2d** [s]

Show that the axially symmetric Green function for Laplace's equation in 2D is

$$G(\underline{x}, \underline{x}') = \frac{1}{2\pi}\ln|\underline{x} - \underline{x}'|$$

9.9 **Laplace's equation in the quarter plane**[s]

The function $u(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{for} \quad 0 < x < \infty \quad \text{and} \quad 0 < y < \infty$$

The boundary conditions are

$$\begin{aligned} u(x, 0) &= 0 & \qquad u_x(0, y) &= -Q\theta(b - y) \\ u_x \text{ and } u_y &\to 0 & \text{as} \quad x &\text{ or } y \to \infty \end{aligned}$$

By taking the Fourier sine transform with respect to y, obtain the solution

$$u(x, y) = \frac{2Q}{\pi}\int_0^\infty \frac{1 - \cos(kb)}{k^2} e^{-kx}\sin(ky)\, dk$$

9.10 **The method of images** [s]

a) Find the position and magnitude of the image charges appropriate to the following problems:-

(i) A point charge q at distance $d - R > 0$ away from the surface of a conducting sphere of radius R which is earthed ($\phi = 0$ on sphere) (ii)* The same as part (i) but with the sphere isolated (carries zero net charge)

(iii) A dipole $\underline{p}$ at distance d from a conducting plane, with $\underline{p}$ at an angle θ to the normal to the plane.

b)

(i) For the 1d heat conduction equation determine the Green function.

(ii) For the case of the semi-infinite rod (see 9.5) where the boundary is kept at fixed temperature $T(0,t) = 0$, use the method of images to construct the solution for an initial delta function source at $x = a$, $t = 0$.