EM 3 Section 1: Revision: Whistlestop tour of Vector Calculus

You will have met vector calculus in second year mathematics courses. This year we shall see the true utility and power of vector calculus in formulating electrostatics. You need to revise *div, grad, curl* and *line, surface and volume integrals*. The following highlights some keypoints but does not replace your second year notes.

1. 1. Gradient

The gradient operator ("grad") acting on a scalar field $f(\underline{r})$ is a vector which in Cartesian Co-ordinates (x,y,z) reads

$$\boxed{\underline{\nabla} f = \frac{\partial f}{\partial x}\underline{e}_x + \frac{\partial f}{\partial y}\underline{e}_y + \frac{\partial f}{\partial z}\underline{e}_z} \tag{1}$$

Important things to remember:

- $\underline{\nabla} f$ is a **vector** quantity (vectors either underlined or boldface in these notes)
- $\underline{\nabla} f$ points in the direction of maximum increase of f
- $\underline{\nabla} f$ is perpendicular to the level surfaces of f
- For a small change of position $\underline{\mathrm{d}r}$ the change in f is $\mathrm{d}f = \underline{\nabla} f \cdot \underline{\mathrm{d}r}$
- The line integral $\displaystyle\int_A^B \underline{\nabla} f \cdot \underline{\mathrm{d}l} = f_B - f_A$ is independent of the path from A to B

Simple example to be memorised $\boxed{\underline{\nabla} r = \underline{\hat{r}}.}$

Remark Often due to the symmetry of the problem it is convenient to consider other co-ordinate systems such as *spherical polar coordinates* which comprise (r, ϕ, θ) or *cylindrical polar coordinates* which comprise (ρ, ϕ, z) (you should remind yourselves of these co-ordinate systems). In these systems the expression for the gradient (and the other operations below) look more complicated e.g. in spherical polars

$$\underline{\nabla} f = \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{e}_\phi$$

(where $\underline{e}_r = \underline{\hat{r}}$). You don't need to remember the general formulae—you can look them up. But when the system has a *spherical symmetry* $f = f(r)$ (no θ or ϕ dependence) the gradient is simply $\underline{\nabla} f = \frac{\partial f}{\partial r}\underline{e}_r$ and this should be remembered.

This is consistent with the *chain rule* which states

$$\boxed{\underline{\nabla} f(r) = \frac{\mathrm{d}f}{\mathrm{d}r}\underline{\nabla} r = \frac{\mathrm{d}f}{\mathrm{d}r}\underline{\hat{r}}} \tag{2}$$

Important example: $\boxed{\underline{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\underline{\hat{r}}}$

1. 2. Divergence and the Divergence Theorem

The divergence ("div") is a scalar product $\underline{\nabla}\cdot$ of the gradient operator with a vector field $\underline{K}$. In Cartesians it reads

$$\boxed{\underline{\nabla} \cdot \underline{K} = \frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} + \frac{\partial K_z}{\partial z}} \tag{3}$$

The divergence represents the rate with which flux lines of the vector field $\underline{K}$ are converging towards sinks (negative divergence), or diverging from sources (positive divergence).

Simple example: $\underline{\nabla} \cdot \underline{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

The **divergence theorem** (to be memorised) states that:

$$\boxed{\oint_A \underline{K} \cdot \mathrm{d}\underline{S} = \int_V \underline{\nabla} \cdot \underline{K} \, \mathrm{d}V} \tag{4}$$

where A is a closed surface enclosing a volume V, $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is a volume element (sometimes written d^3r), and $\mathrm{d}\underline{S} = \hat{\underline{n}}\mathrm{d}S$ is a vector element of area (normal to the surface).

Thus the divergence theorem relates an integral over a closed surface to an integral over the volume enclosed

This theorem holds for any vector field $\underline{K}$ and any closed surface A.

1. 3. Curl and Stokes' Theorem

The curl operator is a vector product of the gradient operator $\underline{\nabla}\times$ with a vector field $\underline{K}$:

$$\underline{\nabla} \times \underline{K} = \left[\frac{\partial K_z}{\partial y} - \frac{\partial K_y}{\partial z}\right] \underline{e_x} + \left[\frac{\partial K_x}{\partial z} - \frac{\partial K_z}{\partial x}\right] \underline{e_y} + \left[\frac{\partial K_y}{\partial x} - \frac{\partial K_x}{\partial y}\right] \underline{e_z} \tag{5}$$

or

$$\boxed{\underline{\nabla} \times \underline{K} = \begin{vmatrix} \underline{e_x} & \underline{e_y} & \underline{e_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ K_x & K_y & K_z \end{vmatrix}} \tag{6}$$

The curl $\underline{\nabla} \times \underline{K}(\underline{r})$ measures how vector field $\underline{K}(\underline{r})$ rotates in space near some point $\underline{r}$. The curl is a vector and its direction is the axis of rotation by the right-hand rule and the magnitude of curl is the magnitude of the rotation

Simple example to be memorised: $\boxed{\underline{\nabla} \times \underline{r} = 0}$

Stokes's theorem (to be memorised) states that:

$$\boxed{\oint_C \underline{K} \cdot \mathrm{d}\underline{l} = \int_A \underline{\nabla} \times \underline{K} \cdot \mathrm{d}\underline{S}} \tag{7}$$

where C is a closed contour bounding a surface A.

Thus Stokes theorem relates a line integral around a closed curve to a surface integral over any open surface bounded by that curve.

This theorem holds for any vector field $\underline{K}$ and any closed curve C.

1. 4. Laplacian

The Laplacian of a scalar field is a scalar defined as

$$\nabla^2 f = \underline{\nabla} \cdot (\underline{\nabla} f) \tag{8}$$

and reads in Cartesians

$$\boxed{\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}} \tag{9}$$

1. 5. Useful Identities

These are best proved by suffix notation.

First, there are various product identities. Generally these are as you'd expect,

1. $\underline{\nabla}(\phi f) = \phi \underline{\nabla} f + (\underline{\nabla}\phi) f$
2. $\underline{\nabla} \cdot (\phi\, \underline{A}) = \phi\, \underline{\nabla} \cdot \underline{A} + \underline{A} \cdot \underline{\nabla}\phi$
3. $\underline{\nabla} \times (\phi\, \underline{A}) = \phi\, (\underline{\nabla} \times \underline{A}) + (\underline{\nabla}\phi) \times \underline{A}$

You should be able to write these down.

Others are less obvious and do not need to be memorised:

4. $\underline{\nabla}\,(\underline{A} \cdot \underline{B}) = (\underline{A} \cdot \underline{\nabla})\, \underline{B} + (\underline{B} \cdot \underline{\nabla})\underline{A} + \underline{A} \times (\underline{\nabla} \times \underline{B}) + \underline{B} \times (\underline{\nabla} \times \underline{A})$
5. $\underline{\nabla} \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot (\underline{\nabla} \times \underline{A}) - \underline{A} \cdot (\underline{\nabla} \times \underline{B})$
6. $\underline{\nabla} \times (\underline{A} \times \underline{B}) = \underline{A}\,(\underline{\nabla} \cdot \underline{B}) - \underline{B}\,(\underline{\nabla} \cdot \underline{A}) + (\underline{B} \cdot \underline{\nabla})\, \underline{A} - (\underline{A} \cdot \underline{\nabla})\, \underline{B}$

Secondly, there are some simple identities (involving two grads) that prove fundamental to Electromagnetism

- "curl grad = 0"

$$\boxed{\underline{\nabla} \times (\underline{\nabla} f) = 0} \tag{10}$$

  where $f(\underline{r})$ is any scalar field.

- "div curl = 0"

$$\boxed{\underline{\nabla} \cdot (\underline{\nabla} \times \underline{K}) = 0} \tag{11}$$

  where $\underline{K}(\underline{r})$ is any vector field.

- "curl curl = grad div - delsquared"

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{K}) = \underline{\nabla}(\underline{\nabla}.\underline{K}) - \nabla^2 \underline{K} \tag{12}$$

  Note that:

$$\nabla^2 K_x = \frac{\partial^2 K_x}{\partial x^2} + \frac{\partial^2 K_x}{\partial y^2} + \frac{\partial^2 K_x}{\partial z^2}$$

The first two (10,11) are crucial to remember now. The third will become important later on.

*1. 6. * 3d Taylor expansion*

As noted above the change in f due to a small change of position $\underline{\mathrm{d}r}$ is $\mathrm{d}f = \underline{\nabla} f \cdot \underline{\mathrm{d}r}$

This is actually the first term in the 3d Taylor expansion about a point $\underline{r}'$ which may be neatly written

$$f(\underline{r}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[(\underline{r} - \underline{r}') \cdot \underline{\nabla} \right]^n f(\underline{r})|_{\underline{r}=\underline{r}'} \tag{13}$$

$$= f(\underline{r}') + \sum_{i=1}^{3} (x_i - x_i') \left. \frac{\partial f(\underline{r})}{\partial x_i} \right|_{\underline{r}=\underline{r}'} + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} (x_i - x_i')(x_j - x_j') \left. \frac{\partial^2 f(\underline{r})}{\partial x_j \partial x_i} \right|_{\underline{r}=\underline{r}'} \cdots \tag{14}$$

Often the first two terms $f(\underline{r}) \simeq f(\underline{r}') + (\underline{r} - \underline{r}') \cdot \underline{\nabla} f(\underline{r})|_{\underline{r}=\underline{r}'}$ is all we require.

1. 7. Important Theorem

The following three statements concerning a vector field $\underline{F}$ over some region in space are equivalent

1. $\underline{\nabla} \times \underline{F} = 0$ *the vector field is irrotational*
2. $\underline{F} = \underline{\nabla}\phi$ *the vector field may be written as the gradient of a scalar field*
3. the line integral of the field $\int_A^B \underline{F} \cdot \underline{\mathrm{d}l}$ *is independent of the path from A to B*;
   a consequence is $\oint_C \underline{F} \cdot \underline{\mathrm{d}l} = 0$ for any closed curve C

You should remind yourselves of how each implies the other
e.g. Stokes theorem gives 1. $\Leftrightarrow$ 3.

This theorem is the heart of electrostatics.