EM 3 Section 11: Inductance

11. 1. Examples of Induction

As we saw last lecture an emf can be induced by changing the area of a current loop in a magnetic field or moving a current loop into or out of a magnetic field.

Here we consider some common examples of rotation of a current loop about its axis in a uniform magnetic field.

AC generator

A generator has a coil of area A rotating about its diameter in a uniform magnetic field with angular velocity ω: In this case it is only the angle between the field and the loop that is varying:

Figure 1: AC generator

$$\Phi_B = AB\cos\omega t \tag{1}$$

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -AB\omega\sin\omega t \tag{2}$$

This system generates an alternating current (AC) with frequency ω. The current is $\pi/2$ out of phase with the rotation, so the peak current is obtained when the flux is zero, i.e. when the loop is parallel to the magnetic field. There is zero current when the loop is perpendicular to the field.

Rotating disc of charge

An insulating disc with a uniform surface charge rotates around its axis. There is a uniform magnetic field parallel to the axis of the disc.

The force on an element of charge, q, on the disc at radius, r, is:

$$\underline{\mathrm{d}F} = qvB\underline{e}_r = qr\omega B\underline{e}_r \tag{3}$$

where $\underline{e}_r$ is the radial basis vector on the disc. This magnetic force is equivalent to a *radial* electric field:

$$\underline{E}' = \underline{F}/q = r\omega B\underline{e}_r \tag{4}$$

and there is an induced emf between the centre and outer radius of the disc:

$$\mathcal{E} = \int \underline{E}' \cdot \underline{\mathrm{d}r} = \frac{\omega B a^2}{2} \tag{5}$$

This emf acts outwards to try and move the charge to the outside of the disc. *If the disc were a conductor this would actually happen.*

So what happens to the flux rule for this type of problem? Basically it is not clear if there is any current loop to consider a flux through. Thus, as it stands, the flux rule $\mathcal{E} = -\mathrm{d}\Phi_B/\mathrm{d}t$ only works when there is a fixed current loop.

11. 2. Mutual Inductance

Consider two current loops I_1,I_2 at rest. The current I_1 will lead to a magnetic field $\underline{B}_1$ which will lead to a magnetic flux through loop 2

$$\boxed{\Phi_2 = \int \underline{B}_1 \cdot \underline{\mathrm{d}S}_2 \equiv M_{21} I_1} \tag{6}$$

M_{21} is the **mutual inductance** of the two loops; it relates the flux through loop 2 to the current in loop 1.

Now let us use the vector potential and Stokes' theorem to obtain an explicit form for M_{12}

$$\begin{aligned} \Phi_2 &= \int (\underline{\nabla} \times \underline{A}_1) \cdot \underline{\mathrm{d}S}_2 = \oint_2 \underline{A}_1 \cdot \underline{\mathrm{d}l}_2 \\ &= \frac{\mu_0 I_1}{4\pi} \oint_1 \oint_2 \frac{\underline{\mathrm{d}l}_1 \cdot \underline{\mathrm{d}l}_2}{|\underline{r}_1 - \underline{r}_2|} \end{aligned}$$

where we have used the formula for the vector potential from section 9 equation (10). Thus

$$\mathrm{M}_{12} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 \frac{\underline{\mathrm{d}l}_1 \cdot \underline{\mathrm{d}l}_2}{|\underline{r}_1 - \underline{r}_2|} \tag{7}$$

where the integrals are taken round both current loops. This is known as the Neumann formula but it is not very useful for most practical applications. What it does reveal is that

$$\boxed{\mathrm{M}_{12} = \mathrm{M}_{21} = \mathrm{M}} \tag{8}$$

which is a remarkable result i.e the flux through 1 when there is current I in 2 is the same as the flux through 2 when there is current I in 1 whatever the geometry of the loops! The relative geometry of the two conductors enters through M which is a purely geometric quantity (a double integral around the loops)

Now let us introduce time dependence and vary the currentI_1 in 1. The changing flux through 2 then gives rise to an emf

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_2}{\mathrm{d}t} = -M\frac{\mathrm{d}I_1}{\mathrm{d}t}$$

By Lenz's law this emf *opposes* the change in current.

11. 3. Self-Inductance

The above discussion similarly applies to the source loop itself i.e. a changing current in a loop induces a "back emf" which opposes the change in current.

The self-inductance of the loop, L, is defined as the ratio of the induced emf to the current change:

$$\mathcal{E} = -\mathrm{L}\frac{\mathrm{dI}}{\mathrm{d}t} \tag{9}$$

It can also be written as:

$$\boxed{\Phi_B = LI} \tag{10}$$

The unit of inductance is the Henry (H), which is 1 Vs/A.

Inductance of a Solenoid

For a long solenoid (length l, radius a, with n loops per unit length) there is a uniform magnetic field along the axis of the solenoid:

$$B_z = \mu_0 n\mathrm{I} \tag{11}$$

This result can be shown using Ampère's Law (see tutorial 5.1).

The flux through *all nl* loops is:

$$\Phi_B = \int_A \underline{B}\cdot \mathrm{d}\underline{S} = \mu_0 n\mathrm{I}\pi a^2 nl$$

and the self-inductance of the solenoid is:

$$\mathrm{L} = \mu_0 n^2 \pi a^2 l = \mu_0 n^2 V \tag{12}$$

11. 4. Energy Stored in Inductors

The work done to create a current in a loop *against* the induced emf is related to the self-inductance L:

$$\frac{\mathrm{d}U_M}{\mathrm{d}t} = -\mathcal{E}\mathrm{I} = \mathrm{LI}\frac{\mathrm{dI}}{\mathrm{d}t}$$

Integrating this with respect to time gives:

$$\boxed{U_M = \frac{1}{2}\mathrm{LI}^2} \tag{13}$$

For two coils with a mutual inductance:

$$U_M = \frac{1}{2}\mathrm{L}_1{\mathrm{I}_1}^2 + \frac{1}{2}\mathrm{L}_2{\mathrm{I}_2}^2 + M_{12}\mathrm{I}_1\mathrm{I}_2$$

Example of solenoid For a long solenoid the self inductance and magnetic field are:

$$L = \mu_0 n^2 \pi a^2 l \qquad \underline{B} = \mu_0 n\mathrm{I}\underline{e}_z \quad \text{inside solenoid}$$

The energy stored in the solenoid is:

$$U_M = \frac{1}{2}\mu_0 n^2 \mathrm{I}^2 \pi a^2 l = \frac{1}{2\mu_0}|\underline{B}|^2 \pi a^2 l$$

This can be written in terms of the energy *density* associated with the magnetic field:

$$u_M = \frac{|\underline{B}|^2}{2\mu_0}$$

Note that this treatment of the energy density of a magnetic field in an inductor is very similar to the treatment of the energy density of an electric field in a capacitor.

We can write the result (13) in a form that uses the magnetic vector potential and the current density. As before the flux is given by

$$\Phi_B = \int (\underline{\nabla}\times\underline{A})\cdot \mathrm{d}\underline{S} = \oint \underline{A}\cdot \mathrm{d}\underline{l}$$

Thus using the definition of L

$$LI = \oint \underline{A}\cdot \mathrm{d}\underline{l}$$

and we find for a current loop that

$$U_M = \frac{I}{2}\oint \underline{A}\cdot \mathrm{d}\underline{l} = \frac{1}{2}\oint \underline{A}\cdot \underline{I}\mathrm{d}l$$

The generalisation to volume currents is

$$\boxed{U_M = \frac{1}{2}\int \underline{A}\cdot \underline{J}\,\mathrm{d}V} \tag{14}$$

We can develop (14) further by using Ampère's law and a product rule from lecture 1

$$\begin{aligned}
\mu_0 \underline{A}\cdot\underline{J} &= \underline{A}\cdot(\underline{\nabla}\times\underline{B}) \\
&= \underline{B}\cdot(\underline{\nabla}\times\underline{A}) - \underline{\nabla}\cdot(\underline{A}\times\underline{B}) \\
&= \underline{B}\cdot\underline{B} - \underline{\nabla}\cdot(\underline{A}\times\underline{B})
\end{aligned}$$

Consequently

$$\begin{aligned}
U_M &= \frac{1}{2\mu_0}\left[\int B^2 \mathrm{d}V - \int \underline{\nabla}\cdot(\underline{A}\times\underline{B})\,\mathrm{d}V\right] \\
&= \frac{1}{2\mu_0}\left[\int B^2 \mathrm{d}V - \oint_S (\underline{A}\times\underline{B})\cdot \mathrm{d}\underline{S}\right]
\end{aligned}$$

The second integral is a boundary term which vanishes when we take the volume over all space and assumes $\underline{B}$ vanishes at ∞, therefore

$$\boxed{U_M = \frac{1}{2\mu_0}\int_{\text{allspace}} |\underline{B}|^2 \mathrm{d}V} \tag{15}$$

In a similar way to the electrostatic energy U_E, we can think of the magnetic energy being stored either in the (localised) current distribution (14) or throughout all space in the magnetic field (15).