EM 3 Section 13: Description of Electromagnetic Waves

13. 1. Recap of wave equations

Let us recall (see Mathematics for Physics 4 and Physics 2A) the wave equation in 1d (i.e. one spatial dimension x and one time dimension t) for a scalar field u

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}} \tag{1}$$

Now, as can readily be checked by substitution into (1), the general solution is *any* function f of the form

$$u(x,t) = f(kx - \omega t) \tag{2}$$

where the wave velocity c is given by

$$c = \frac{w}{k} \tag{3}$$

A convenient solution of special interest is

$$f = A \exp i(kx - \omega t) = A\cos(kx - \omega t) + iA\sin(kx - \omega t) \tag{4}$$

These are sinusoidal waves and A is the constant amplitude (which may be complex) N.B. the real (cosine) and imaginary (sine) parts are independent solutions. Moreover it is a **monochromatic wave** since there is a single angular frequency ω. These are the basis of Fourier methods where we build up waves of arbitrary shape by superposition of sines and cosines

If (for physical reasons) we want to get a real solution from (4) we simply take the real part

$$\begin{aligned} u(x,t) &= \mathrm{Re}\left[A \exp i(kx - \omega t)\right] \\ &= \mathrm{Re}A\, \cos(kx - \omega t) - \mathrm{Im}A\, \sin(kx - \omega t) \end{aligned} \tag{5}$$

Important things to remember are : k is the wavenumber; the angular frequency is $\omega = 2\pi\nu$ where ν is the frequency; the wavelength is $\lambda = 2\pi/k$; the whole wave proceeds to the right with speed c, but at any fixed x the wave oscillates with period $T = 2\pi/\omega = 2\pi/kc$.

The 1d equation (1) generalises easily to 3d

$$\boxed{\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}} \tag{6}$$

where the second derivative w.r.t. x has been replaced by the Laplacian operator.

The solution (2) generalises to

$$u(\underline{r}, t) = f(\underline{k}\cdot\underline{r} - \omega t) \tag{7}$$

where the **wavevector** $\underline{k} = (k_x, k_y, k_z)$ and the velocity is again

$$c = \frac{w}{k} \tag{8}$$

where $k = |\underline{k}|$. We can also write (7) as

$$u(\underline{r}, t) = g(\underline{\hat{n}} \cdot \underline{r} - ct) \tag{9}$$

where $\underline{\hat{n}}$ is the unit vector in the direction of $\underline{k}$.

13. 2. Plane Waves

The generalisation of the 1d sinusoidal solution (4) is to the 3d **plane wave** solution

$$u(\underline{r}, t) = A \exp i(\underline{k} \cdot \underline{r} - \omega t) \tag{10}$$

The is called a plane wave because it takes the same (complex) value whenever

$$\underline{k} \cdot \underline{r} = \omega t + \text{constant} \tag{11}$$

which at any fixed t is the equation of a *plane* with normal in the $\underline{k}$ direction.
To see that (10) is a solution to (12) note that

$$\begin{aligned} \underline{\nabla} \exp i\underline{k} \cdot \underline{r} &= \left[\underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} + \underline{e}_z \frac{\partial}{\partial x} \right] \exp i(k_x x + k_y y + k_z z) \\ &= i\underline{k} \exp i\underline{k} \cdot \underline{r} \end{aligned}$$

and

$$\nabla^2 \exp i\underline{k} \cdot \underline{r} = \underline{\nabla} \cdot \underline{\nabla} \exp i\underline{k} \cdot \underline{r} \quad = \quad i\underline{\nabla} \cdot (\underline{k} \exp i\underline{k} \cdot \underline{r}) = i\underline{k} \cdot \underline{\nabla} \exp i\underline{k} \cdot \underline{r} = -k^2 \exp i\underline{k} \cdot \underline{r}$$

where we used the product identity $\underline{\nabla} \cdot (\underline{k} f) = f \underline{\nabla} \cdot \underline{k} + \underline{k} \cdot \underline{\nabla} f = \underline{k} \cdot \underline{\nabla} f$ since $\underline{k}$ is constant.
Also

$$\frac{\partial^2 \exp i(\underline{k} \cdot \underline{r} - \omega t)}{\partial t^2} = -\omega^2 \exp i(\underline{k} \cdot \underline{r} - \omega t)$$

Finally we can generalise to the 3d wave equation for a *vector field* $\underline{F}$

$$\boxed{\nabla^2 \underline{F} = \frac{1}{c^2} \frac{\partial^2 \underline{F}}{\partial t^2}} \tag{12}$$

for which a plane wave solution is

$$\underline{F} = \underline{F}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t) \tag{13}$$

where $\underline{F}_0$ is a constant (complex) vector. The key things to remember with this plane wave solution are

$$\underline{\nabla} \cdot \underline{F} = i\underline{k} \cdot \underline{F} \tag{14}$$
$$\underline{\nabla} \times \underline{F} = i\underline{k} \times \underline{F} \tag{15}$$
$$\nabla^2 \underline{F} = -k^2 \underline{F} \tag{16}$$

13. 3. Electromagnetic Plane Waves

Previously we saw that *in vacuo* Maxwell's equations with $\rho = 0$, $\underline{J} = 0$ read

$$\begin{aligned} \underline{\nabla} \cdot \underline{E} &= 0 & (17)\\ \underline{\nabla} \cdot \underline{B} &= 0 & (18)\\ \underline{\nabla} \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} & (19)\\ \underline{\nabla} \times \underline{B} &= \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t} & (20) \end{aligned}$$

and reduce to the decoupled wave equations

$$\begin{aligned} \nabla^2 \underline{E} &= \epsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2} & (21)\\ \nabla^2 \underline{B} &= \epsilon_0 \mu_0 \frac{\partial^2 \underline{B}}{\partial t^2} & (22) \end{aligned}$$

Clearly we have plane solutions

$$\underline{E} = \underline{E}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t) \quad \underline{B} = \underline{B}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t) \tag{23}$$

moving at the speed of light $c = \omega/k = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$. However Maxwell's equations imply more constraints on our plane wave solutions. First MI, MII (17,18) imply

$$i\underline{k} \cdot \underline{E}_0 = 0 \qquad i\underline{k} \cdot \underline{B}_0 = 0$$

i.e. $\underline{E}_0$ and $\underline{B}_0$ and hence $\underline{E}$ and $\underline{B}$ are *perpendicular* to the direction of propagation $\underline{k}$. That is, the wave is **transverse**.

It is conventional to take the direction of propagation $\underline{k}$ in the $\underline{e}_z$ direction;

$$\underline{k} = k\underline{e}_z \tag{24}$$

therefore $\underline{E}_0$ and $\underline{B}_0$ lie in the x–y plane. Substituting (23) in MIII we find

$$i\underline{k} \times \underline{E}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t) \quad = i\omega \underline{B}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t)$$

or more compactly

$$\underline{B}_0 = \frac{k}{\omega}(\underline{e}_z \times \underline{E}_0) \tag{25}$$

Now since $\underline{e}_z$ and $\underline{E}_0$ are orthogonal we can take magnitudes

$$|\underline{B}_0| = \frac{k}{\omega}|\underline{E}_0| \tag{26}$$

Now we should choose the directions of $\underline{B}_0$ and $\underline{E}_0$. (25) tells us that **the magnetic field is perpendicular to the electric field, and both are perpendicular to the direction of propagation of the wave**.

Polarisation states can be defined in various ways:

- Linearly (or plane) polarized - direction of $\underline{E}$ is fixed. There are two orthogonal plane polarisation states with $\underline{E}$ in x or y direction.
- Circularly polarized - direction of $\underline{E}$ rotates clockwise or anticlockwise around the z axis in the x–y plane.

An unpolarised electromagnetic wave is a random mixture of polarisation. So $\underline{E}$ has a random directions as a function of z.

13. 4. Linear (Plane) Polarisation

Let us first consider the case where $\underline{B}_0$ and $\underline{E}_0$ are *real*. Then, since they lie in the x–y plane it is conventional to take $\underline{E}_0 = E_0\underline{e}_x$ and $\underline{B}_0 = B_0\underline{e}_y$. This is referred to as *linear polarisation in the x direction i.e. the electric field is always in the x direction and magnetic field is always in the y direction and $\underline{k}$ is in the z direction as usual.* Polarisation in the y direction would have $\underline{E}_0 = E_0\underline{e}_y$, $\underline{B}_0 = -B_0\underline{e}_x$. More generally we can take $\underline{E}_0 = E_0\underline{\hat{n}}$

Figure 1: Plane polarisation in x direction *(Griffiths fig 9.10)*

$$\underline{E}_0 \cdot \underline{e}_x = E_0\underline{\hat{n}} \cdot \underline{e}_x = E_0 \cos\theta$$

where $\underline{\hat{n}}$ is the polarisation vector and θ is the polarisation angle.

13. 5. Circular Polarisation

Now consider taking $\underline{E}_0$ as a *complex* vector

$$\underline{E}_0 = \frac{E_0}{\sqrt{2}}(\underline{e}_x \pm i\underline{e}_y)\mathrm{e}^{i\phi} \tag{27}$$

Then we find that the real part of $\underline{E}$ is given by

$$\mathrm{Re}\,\underline{E} = \frac{E_0}{\sqrt{2}}\left[\underline{e}_x \cos(\underline{k}\cdot\underline{r} - \omega t + \phi) \mp \underline{e}_y \sin(\underline{k}\cdot\underline{r} - \omega t + \phi)\right] \tag{28}$$

The *minus* sign in (27) implies that the polarisation vector rotates anticlockwise about the $\underline{e}_z$: i.e. at time $\omega t = \underline{k}\cdot\underline{r} + \phi$, Re$\underline{E}$ is in the $\underline{e}_x$ direction but as time increases the polarisation vector rotates towards $-\underline{e}_y$. This also referred to left circular polarisation or positive helicity

Likewise the *plus* sign in (27) implies that the polarisation vector rotates clockwise about the $\underline{e}_z$ This is referred to as right circular polarisation or negative helicity.