

EM 3 Section 15: Dielectric Materials

15. 1. Overview

So far we have developed Maxwell's equations and they offer a complete and general description of electrodynamics. However the input we have to make is to define the charge and current densities ρ and \underline{J} with *microscopic precision*. In the real world (i.e. not in vacuo) this would be a huge task as materials are made up of atoms/molecules which all contain charge distributions and currents (through electronic orbits). This is the atomic level of description.

Instead we want to develop a *macroscopic description* of materials in terms of smoothly varying quantities which are averaged in some way: these turn out to be the density ρ_f and current \underline{J}_f of **free charges**. The **bound charges** which are bound up in the atomic structure are dealt with by defining new fields \underline{D} the **Electric Displacement Field** and \underline{H} the **Auxiliary (magnetic) Field**. Then we end up with a complementary *macroscopic form of Maxwell's equations* which is a nice example of an *effective theory* coming from the microscopic theory. Although it may seem annoying to have to learn a second set of Maxwell's equations, they are in some ways simpler than the microscopic ones.

15. 2. Dielectric Materials

Roughly speaking we can classify materials as conductors or dielectrics (insulators). A perfect conductor will have an 'unlimited' supply of free charges whereas at the other extreme a perfect dielectric will have no free charges and instead all charges are bound up in atoms/molecules.

Figure 1: Polarization of Dipoles in a Dielectric

Let us consider the effect of an electric field on a dielectric. The field will induce a dipole moment in two ways

- the charge distribution of some atoms/molecules is distorted
- already polar molecules (e.g. H_2O) will tend to align with the external field (rotation)

These effects *polarize* the material and result in an induced dipole moment for each atom

$$\langle \underline{p}_{atom} \rangle = \alpha \underline{E} \quad (1)$$

where α is the atomic polarizability. We take an average in (1) as an atom's dipole moment will not be constant due to thermal fluctuations. All these atomic dipole moments give rise to the dipole moment per unit volume \underline{P} or **Polarization**. We define the polarization field \underline{P} through the net dipole moment $d\underline{p}$ in a small volume dV

$$\underline{d}p = \underline{P} dV \quad (2)$$

thus \underline{P} is dipole moment per unit volume. We can relate it to the atomic dipole moment $\langle \underline{p}_{atom} \rangle$ through

$$\underline{P} = n \langle \underline{p}_{atom} \rangle$$

where n is the number of atoms per unit volume.

Let us now consider the field *due* to the polarized molecules. Recall that for a single dipole at \underline{r}' the potential at \underline{r} is

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0} \frac{(\underline{r} - \underline{r}') \cdot \underline{p}}{|\underline{r} - \underline{r}'|^3}$$

This generalises by superposition to the potential due to the Polarization field $\underline{P}(\underline{r}')$

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{(\underline{r} - \underline{r}') \cdot \underline{P}(\underline{r}')}{|\underline{r} - \underline{r}'|^3} dV'$$

We now note a usual identity but this time for the gradient wrt the primed coordinates

$$\underline{\nabla}' \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) = \frac{(\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3}$$

Then we perform 'integration by parts' using the divergence theorem

$$\begin{aligned} V(\underline{r}) &= \frac{1}{4\pi\epsilon_0} \int_V \underline{P}(\underline{r}') \cdot \underline{\nabla}' \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) dV' \\ &= \frac{1}{4\pi\epsilon_0} \left[\int_V \underline{\nabla}' \cdot \left(\frac{\underline{P}}{|\underline{r} - \underline{r}'|} \right) dV' - \int_V \frac{1}{|\underline{r} - \underline{r}'|} \underline{\nabla}' \cdot \underline{P} dV' \right] \\ &= \frac{1}{4\pi\epsilon_0} \oint_S \frac{\underline{P} \cdot d\underline{S}'}{|\underline{r} - \underline{r}'|} - \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{|\underline{r} - \underline{r}'|} (\underline{\nabla}' \cdot \underline{P}) dV' \end{aligned}$$

Now the first term on the right hand side is equivalent to the potential due to a surface charge distribution on S i.e. $\underline{P} \cdot d\underline{S} \rightarrow \sigma_b dS$ or

$$\sigma_b = \underline{P} \cdot \hat{n} \quad (3)$$

where \hat{n} is normal to the surface. The second term on the lhs is equivalent to the potential due to a volume charge distribution ρ_b which is given by

$$\rho_b = -\underline{\nabla} \cdot \underline{P} \quad (4)$$

The subscript b refers to the fact the charges are bound (to the atoms)

15. 3. Electric displacement vector and Gauss' law in media

We are now in a position to develop Gauss' law in the case of media. The key idea is to divide up the charge distribution into bound and free charges

$$\rho = \rho_b + \rho_f$$

Then Gauss's law (MI) becomes

$$\nabla \cdot \underline{E} = \frac{\rho_f}{\epsilon_0} + \frac{\rho_b}{\epsilon_0} = \frac{\rho_f}{\epsilon_0} - \frac{\nabla \cdot \underline{P}}{\epsilon_0}$$

or

$$\nabla \cdot (\epsilon_0 \underline{E} + \underline{P}) = \rho_f \quad (5)$$

Now let us define the **Electric displacement** as

$$\underline{D} \equiv \epsilon_0 \underline{E} + \underline{P} \quad (6)$$

Gauss' law in media then becomes

$$\nabla \cdot \underline{D} = \rho_f \quad (7)$$

15. 4. Linear Isotropic Homogeneous Media

So far, so good, but at the expense of the introduction of a new field \underline{D} in addition to \underline{E} . However things become simpler when we consider an ideal type of medium which is *linear, isotropic and homogeneous* (LIH).

Isotropic means there is no preferred direction which implies through symmetry that \underline{P} is \parallel to \underline{E} . *Linear* means that the applied \underline{E} field results in a generally small polarization of molecules through distortion and rotation, and we expect a *linear* response to the field

$$\underline{P} = \chi_E \epsilon_0 \underline{E} \quad (8)$$

χ_E (chi) is the susceptibility—large χ_E means a large response to the applied field and the medium is easier to polarize.

Homogeneous means the medium has the same properties at all points in space so that χ_E has no spatial dependence.

Using (8) results in

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P} = \epsilon_0 (1 + \chi_E) \underline{E}$$

$$\text{or} \quad \underline{D} = \epsilon_0 \epsilon_r \underline{E} \quad (9)$$

where $\epsilon_r = 1 + \chi_E$ is the *relative permittivity* (or dielectric constant) of the medium and is a dimensionless constant = 1 for vacuum; for most insulators $\epsilon_r = 1.05 - 1.3$. Some crystals have high ϵ_r , e.g. mica: $\epsilon_r = 7$. For dipolar fluids, e.g. deionized water: $\epsilon_r = 80$.

The important point is: for LIH we have a linear *constitutive relation* (9) between \underline{E} and \underline{D} .

15. 5. Example: Dielectrics in Capacitors

The space between the two plates of a capacitor can be filled with an insulating material rather than with a vacuum. There are induced polarization (bound) charges on the surfaces next to the plates. These change the capacitance in a way that depends on the geometry of the insulator and the plates. For a *parallel plate capacitor*: the electric field is simply

Figure 2: Parallel plate capacitor with dielectric

the superposition of the field from the free charges on the plates and bound charges at the surface of the dielectric

$$\underline{E} = \underline{E}_0 + \underline{E}_P = \frac{1}{\epsilon_0}(\sigma_f - \sigma_b)\hat{n} \quad (10)$$

where \hat{n} is normal to the plates. The electric field \underline{E} as a function of the free charge density on the plates σ_f is *reduced* by the polarization of the dielectric between the plates. N.B. the total free charge on a plate is still $Q = A\sigma_f$. Also the electric displacement turns out to be simply

$$\underline{D} = \sigma_f\hat{n}$$

this can be checked by the modified version of Gauss's Law which gives

$$\oint_S \underline{D} \cdot d\underline{S} = \int_V \rho_f dV = (Q_f)_{enc} \quad (11)$$

Taking a Gaussian pillbox area a straddling a plate one finds that $a|\underline{D}| = a\sigma_f$

The parallel plate capacitance is given in terms of the potential difference V_d , which remains

$$V_d = - \int_1^2 \underline{E} \cdot d\underline{l}$$

When we integrate along the normal from plate 1 to plate 2

$$V_d = Ed = \frac{Dd}{\epsilon_0\epsilon_r}$$

and

$$C = \frac{Q}{Ed} = \frac{A\sigma_f}{Ed} = \frac{AD}{Ed} = \frac{A\epsilon_r\epsilon_0}{d} = \epsilon_r C_0 \quad (12)$$

where C_0 is the capacitance without the dielectric present. For any geometry of capacitor there is an *increase* in the capacitance due to the presence of a dielectric between the plates. Note that it is not necessarily by just a factor ϵ_r — see tutorial.