EM 3 Section 18: Examples of continuity conditions; waves in media

18. 1. Continuity conditions: examples

Example: Inclined dielectric slab

"The electric field $\underline{E}^o$ outside a large dielectric slab of relative permittivity ϵ_r is uniform and at angle θ to the *normal* to the slab. What is the electric field $\underline{E}^i$ inside the slab?".

Figure 1: Similar to Griffiths Fig 4.34

Outside: $\underline{D}^o = \epsilon_0 \underline{E}^o$ inside: $\underline{D}^i = \epsilon_r \epsilon_0 \underline{E}^i$. Let ψ be the angle between the normal to the plane and $\underline{E}^i$

Take the normal to slab in $\underline{e}_z$ direction and tangent in $\underline{e}_x$ direction and write (x, z) components of fields as

$$\begin{aligned} \underline{E}^o &= (E^o \sin\theta, E^o \cos\theta) \qquad \underline{D}^o = \epsilon_0 (E^o \sin\theta, E^o \cos\theta) \\ \underline{E}^i &= (E^i \sin\psi, E^i \cos\psi) \qquad \underline{D}^i = \epsilon_0 \epsilon_r (E^i \sin\psi, E^i \cos\psi) \end{aligned}$$

Now impose b.c.s:

1. $D_n = D_z$ continuous:

$$D_z^i = D_z^o \quad \Rightarrow \quad \epsilon_0 E^o \cos\theta = \epsilon_0 \epsilon_r E^i \cos\psi$$

$$\Rightarrow E^i = \frac{E^o}{\epsilon_r} \frac{\cos\theta}{\cos\psi}$$

2. $E_t = E_x$ continuous:

$$E_x^i = E_x^o \quad \Rightarrow \quad E^o \sin\theta = E^i \sin\psi$$

$$\Rightarrow E^i = E^o \frac{\sin\theta}{\sin\psi}$$

Result:

$$\begin{aligned} \frac{1}{\epsilon_r} \frac{\cos\theta}{\cos\psi} &= \frac{\sin\theta}{\sin\psi} \\ \Rightarrow \quad \psi &= \tan^{-1}(\epsilon_r \tan\theta) \end{aligned}$$

Checks: For $\theta = \pi/2$: $E^i = E^o$, $\psi = \pi/2$ ($\underline{E}$ is purely tangential, continuous)

For $\theta = 0$: $E^i = E^o/\epsilon_r$, $\psi = 0$ ($\underline{D}$ is purely normal, continuous)

Remarks $\underline{D} \parallel \underline{E}$ everywhere; but the angle of *both* is altered within slab. $\underline{E}^i$ is the superposition of uniform $\underline{E}^o$ with that of the polarization charges on surface of slab. Unless $\theta = 0$, as in a parallel plate capacitor, $\underline{D}/\epsilon_0$ is not "the $\underline{E}$ field you would have had" without the slab which would be $\underline{D}^i = \epsilon_0 \epsilon_r \underline{E}^o$

Example: Spherical cavity in dielectric

"A large block of dielectric of relative permittivity $\epsilon_r > 1$ contains a spherical cavity. The $\underline{E}$ field far away from the cavity is uniform, with magnitude E_0. What are $\underline{E}, \underline{D}$ within the cavity?"

Figure 2: Spherical cavity in dielectric - Griffiths Example 4.7

Use spherical polars with origin at the centre of the sphere and take z axis $\parallel \underline{E}_0$. Therefore there is symmetry w.r.t. ϕ.

At the surface of the spherical cavity $\sigma_p = \underline{P} \cdot \hat{\underline{n}}$ where $\hat{\underline{n}}$ is outwards normal of material (inwards normal of sphere). Therefore the field inside is **enhanced** by $\sigma_p(\theta)$.

The charge around the cavity $\sigma_p(\theta)$ forms an effective *dipole*. Outside the field lines are **distorted** locally by $\sigma_p(\theta)$

Try a **uniform** field in z direction within cavity:

$$V(r < a) = -E_{in} z = -E_{in} r \cos\theta \ .$$

Try the uniform field $\underline{E}_0$ plus a **dipole** form outside

$$V(r > a) = -E_0 r \cos\theta + \frac{A \cos\theta}{r^2}$$

where A is a constant to be fixed.

Recall that these two expressions satisfy Laplace's equation away from the boundary where there are no charges. We now just need to satisfy the boundary conditions on the fields.

First recall that $\underline{E} = -\underline{\nabla} V$ and in spherical polars

$$\underline{\nabla} V = \underline{e}_r \frac{\partial V}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \underline{e}_\phi \frac{1}{r \sin\theta}\frac{\partial V}{\partial \phi}$$

At the boundary: E_t continuous requires E_θ continuous at $r = a$:

$$-E_0 a \sin\theta + \frac{A \sin\theta}{a^2} = -E_{in} a \sin\theta$$

D_n continuous requires $D_r = -\epsilon_r \partial V/\partial r$ continuous at $r = a$:

$$\epsilon_r \left(E_0 \cos\theta + \frac{A \cos\theta}{a^3} \right) = E_{in} \cos\theta$$

Combine these

$$E_{in} = \epsilon_r \left(E_0 + \frac{2A}{a^3} \right) = E_0 - \frac{A}{a^3}$$

eliminate A/a^3:

$$E_{in} = E_0 \frac{3\epsilon_r}{1 + 2\epsilon_r}$$

with $\underline{E}_{in} = E_{in}\underline{e}_z$. Then $\underline{D}_{in} = \epsilon_r\epsilon_0\underline{E}_{in} = \epsilon_0\underline{E}_{in}$ (since the cavity has $\epsilon_r = 1$).

Check: $E_{in} > E_0$ if $\epsilon_r > 1$, field inside enhanced.

Uniqueness $\Rightarrow$ problem solved!

This example may be used to derive an approximate formula for atomic polarizability the Clausius Mosotti equation - see tutorial 10.

18. 2. Waves in media

As a first look at waves in media let's consider a *non-conducting* medium with $\rho_f = 0$, $\underline{J}_f = 0$. Let us write the permittivity $\boxed{\epsilon = \epsilon_0\epsilon_r}$ and the permeability $\boxed{\mu = \mu_0\mu_r}$.

As before when we considered waves in vacuo in lecture 13 we can reduce Maxwell's equation to two decoupled wave equations

$$\nabla^2 \underline{E} = \epsilon\mu \frac{\partial^2 \underline{E}}{\partial t^2} \tag{1}$$

$$\nabla^2 \underline{B} = \epsilon\mu \frac{\partial^2 \underline{B}}{\partial t^2} \tag{2}$$

These are precisely the same as in lecture 13 but with ϵ_0 replaced by ϵ and μ_0 replaced by μ.

Clearly we have plane solutions

$$\underline{E} = \underline{E}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t) \qquad \underline{B} = \underline{B}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t) \tag{3}$$

where $k^2 - \mu\epsilon\omega^2 = 0$. Thus the wave speed is

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$$

and recalling $c = \dfrac{1}{\sqrt{\mu_0\epsilon_0}}$

$$\boxed{\frac{v^2}{c^2} = \frac{1}{\mu_r\epsilon_r} \equiv \frac{1}{n^2}} \tag{4}$$

where n is called the refractive index of the medium

As in lecture 13 MI,MII imply

$$i\underline{k} \cdot \underline{E}_0 = 0 \qquad i\underline{k} \cdot \underline{B}_0 = 0$$

i.e. $\underline{E}$ and $\underline{B}$ are *perpendicular* to the direction of propagation $\underline{k}$ and the wave is **transverse**.

This may seem like a trivial generalisation of waves in vacuo but the physics is remarkable— we have managed to deal with all the atoms, atomic dipoles, polarisation etc by wrapping them up into ϵ and μ and the net result is simply to change the velocity of the wave.

18. 3. Waves in conductors

In conductors there is free charge and currents flow in response to an electric field. As we shall see this has a serious effect on the propagation of an EM wave in a conductor.

Let us start with MIV and use the linear relations

$$\underline{B} = \mu \underline{H} \qquad \underline{D} = \epsilon \underline{E}$$

and Ohm's law $\underline{J} = \sigma \underline{E}$

$$\begin{aligned} \underline{\nabla} \times \underline{H} &= \frac{\partial \underline{D}}{\partial t} + \underline{J}_f \\ \rightarrow \underline{\nabla} \times \underline{B} &= \mu\epsilon \frac{\partial \underline{E}}{\partial t} + \mu\sigma \underline{E} \end{aligned}$$

Now as usual MIII yields

$$\frac{\partial}{\partial t}(\underline{\nabla} \times \underline{B}) = -\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \nabla^2 \underline{E} - \underline{\nabla}(\underline{\nabla} \cdot \underline{E})$$

The microscopic M1 reads $\underline{\nabla} \cdot \underline{E} = \rho/\epsilon$. Let us assume a *uniform* charge density so that $\underline{\nabla}\rho = 0$.

Then finally we obtain

$$\boxed{\nabla^2 \underline{E} = \mu\epsilon \frac{\partial^2 \underline{E}}{\partial t^2} + \mu\sigma \frac{\partial \underline{E}}{\partial t}} \qquad (5)$$

we note an additional term on the rhs whose origin is the free current in Maxwell IV. How will this term affect the wave?

A similar calculation (**Exercise**) yields

$$\boxed{\nabla^2 \underline{B} = \mu\epsilon \frac{\partial^2 \underline{B}}{\partial t^2} + \mu\sigma \frac{\partial \underline{B}}{\partial t}} \qquad (6)$$

Let us proceed blindly and bravely by making an ansatz of plane wave moving in the z direction $\underline{E} = \underline{E}_0 \exp i(\tilde{k}z - \omega t)$. When we sub this into (5) we obtain

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

Clearly something has to become complex to solve this!