EM 3 Section 21: Reflection at boundaries: oblique incidence

Last lecture we analysed the case of waves impinging on an interface at normal incidence. Here we consider a general angle of incidence

21. 1. General Angle of Incidence

As before we take an interface between two media to be the x–y plane at $z = 0$: medium 1

Figure 1: Wave at interface between two media *Griffiths fig. 9.14*

is $z < 0$; medium 2 is $z > 0$.

We can take the incident wave vector $\underline{k}_I$ to be in the x–z plane which is then the **plane of incidence**; y out of page

$$
\begin{aligned}
\underline{E}_{inc} &= \underline{E}_I\, e^{i(\underline{k}_I\cdot\underline{r}-\omega t)}\\
\underline{E}_{ref} &= \underline{E}_R\, e^{i(\underline{k}_R\cdot\underline{r}-\omega t)}\\
\underline{E}_{trans} &= \underline{E}_T\, e^{i(\underline{k}_T\cdot\underline{r}-\omega t)}
\end{aligned}
$$

We also have the corresponding magnetic field vectors e.g.

$$\underline{H}_{inc} = \frac{1}{\mu_1 v_1}\hat{\underline{k}} \times \underline{E}_{inc}$$

Now we have to fit the boundary conditions at the interface. First of all we note that all the boundary conditions will be of the form

$$(\quad)\mathrm{e}^{i(\underline{k}_I\cdot\underline{r}-\omega t)} + (\quad)\mathrm{e}^{i(\underline{k}_R\cdot\underline{r}-\omega t)} = (\quad)\mathrm{e}^{i(\underline{k}_T\cdot\underline{r}-\omega t)}$$

So for the boundary conditions to hold for all points on the interface x–y plane we must have the exponential factors (i.e. the phases) equal

$$\Rightarrow \underline{k}_I\cdot\underline{r} = \underline{k}_R\cdot\underline{r} = \underline{k}_T\cdot\underline{r} = \phi = \quad \text{constant} \tag{1}$$

and straightaway we see that $\underline{k}_I$, $\underline{k}_R$, $\underline{k}_T$, all lie in the same plane—the plane of incidence. i.e. none of them has a component in the y direction

Then (1) becomes

$$k_I \sin\theta_I x = k_R \sin\theta_R x = k_T \sin\theta_T x$$

But this must hold for all x and also we know from $k = \omega/v$ that

$$k_I = k_R = \frac{v_2}{v_1}k_T = \frac{n_1}{n_2}k_T$$

which together imply

$$\begin{aligned} \theta_I &= \theta_R & \textbf{angle of incidence equals angle of reflection} \\ n_1 \sin\theta_I &= n_2 \sin\theta_T & \textbf{Snell's Law} \end{aligned}$$

We now have the job of satisfying the boundary conditions (see sections 17,18) which become

$$\begin{aligned} \epsilon_1(\underline{E}_I + \underline{E}_R)_z &= \epsilon_2(\underline{E}_T)_z & (2) \\ (\underline{B}_I + \underline{B}_R)_z &= (\underline{B}_T)_z & (3) \\ (\underline{E}_I + \underline{E}_R)_{x,y} &= \epsilon_2(\underline{E}_T)_{x,y} & (4) \\ (\underline{H}_I + \underline{H}_R)_{x,y} &= (\underline{H}_T)_{x,y} & (5) \end{aligned}$$

The final two equations are both pairs of equation for the two transverse x,y components.

Polarisation Effects

The reflection and transmission coefficients r, t depend on the polarisation state of the incident beam. There are two basic polarisation states

A. $\underline{E}_I$ in plane of incidence ($\underline{E}_I$ has no y component and $\underline{H}_I$ along $\underline{e}_y$)

B. $\underline{H}_I$ in plane of incidence ($\underline{H}_I$ has no y component and $\underline{E}_I$ along $\underline{e}_y$)

Other polarisation states can be decomposed into A+B by superposition. We will only work out **CASE A:** $\underline{E}$ in plane of incidence

Figure 2: Wave at interface between two media *Griffiths fig. 9.15*

Clearly (3) is automatically satisfied as $\underline{B}$ has no z component. It turns out (as you can check) that (5) does not give any additional information to (2) and (4)

Thus noting sign of E_z: $-$ for I,T but $+$ for R

$$\begin{aligned} \underline{E}_{inc} &= E_I \; (\underline{e}_x \cos\theta_I - \underline{e}_z \sin\theta_I)\, \mathrm{e}^{i(\phi - \omega t)} \\ \underline{E}_{ref} &= E_R \; (\underline{e}_x \cos\theta_I + \underline{e}_z \sin\theta_I)\, \mathrm{e}^{i(\phi - \omega t)} \\ \underline{E}_{trans} &= E_T \; (\underline{e}_x \cos\theta_T - \underline{e}_z \sin\theta_T)\, \mathrm{e}^{i(\phi - \omega t)} \end{aligned}$$

Then condition (2) $\Rightarrow$ $\quad \epsilon_1(-E_I + E_R)\sin\theta_I = -\epsilon_2 E_T \sin\theta_T$

and condition (4)$\Rightarrow$ $\quad (E_I + E_R)\cos\theta_I = E_T \cos\theta_T$

2 equations in 2 unknowns, solve for E_T, E_R: We define as before

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \left(\frac{\mu_1 \epsilon_2}{\epsilon_1 \mu_2}\right)^{1/2}$$

and also

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

then we find

$$\boxed{r \equiv \frac{E_R}{E_I} = \frac{\alpha - \beta}{\alpha + \beta} \qquad t \equiv \frac{E_T}{E_I} = \frac{2}{\alpha + \beta}} \tag{6}$$

which can also be written as

$$r = \frac{Z_2 \cos\theta_T - Z_1 \cos\theta_I}{Z_2 \cos\theta_T + Z_1 \cos\theta_I} \qquad t = \frac{2 Z_2 \cos\theta_I}{Z_2 \cos\theta_T + Z_1 \cos\theta_I} \tag{7}$$

where Z_i is the usual impedance. For nonmagnetic dielectrics $\mu_1 = \mu_2 = \mu_0$, $Z_1 = Z_{vac}/n_1$, $Z_2 = Z_{vac}/n_2 \Rightarrow$

$$r = \frac{n_1 \cos\theta_T - n_2 \cos\theta_I}{n_1 \cos\theta_T + n_2 \cos\theta_I}$$

use Snell's law to eliminate n's:

$$\boxed{r = \frac{\sin 2\theta_T - \sin 2\theta_I}{\sin 2\theta_T + \sin 2\theta_I}} \tag{8}$$

Fresnel Formula for case A ($\underline{E}$ in plane of incidence)

21. 2. Brewster's Angle

An interesting consequence of Fresnel equations (6) is that $r = 0$ when $\alpha = \beta$. This occurs at special angle of incidence know as **Brewster's angle** $\theta_I = \theta_B$

$$\left(1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B\right)^{1/2} = \beta \cos\theta_B \tag{9}$$

$$\Rightarrow 1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B = \beta^2 (1 - \sin^2\theta_B) \tag{10}$$

and finally this gives

$$\boxed{\sin^2\theta_B = \frac{1 - \beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}} \tag{11}$$

In the typical case $\mu_1 = \mu_2$ we have $\beta = n_2/n_1$ and one can show that

$$\boxed{\tan\theta_B = \frac{n_2}{n_1}} \tag{12}$$

($\theta_B \simeq 50°$ for water/air)

N.B. Brewster's angle only exists for case A: in case B there is no such effect.

Brewster angle microscopy: Shine 'case-A light' on clean surface at $\theta_I = \theta_B$: **no** reflected ray

Now adsorb thin layer of another material: reflected ray caused **solely** by film $\Rightarrow$ sensitive probe of film structure

Polarisation by reflection: Unpolarised light source = random superposition of waves with E in plane of incidence (case A) and transverse to it (case B)

Near to Brewster's angle reflected ray is **almost all polarised**

One can eliminate reflected ray (glare) with polaroid filter which cuts out one plane of polarised light; basis of polaroid sunspecs etc.

21. 3. Total Internal Reflection

Choose $n_1 > n_2$ (e.g. wave leaving dielectric into vacuum) then $\theta_T > \theta_I$; $\theta_T = 90°$ at $\theta_I = \theta_C$. Snell: $\sin\theta_C = n_2/n_1$. For $\theta_I > \theta_C$: we have **Total Internal Reflection**

Figure 3: Total internal reflection *Griffiths fig 9.28*

Evanescent Waves

To see what is happening for $\theta_I > \theta_C$, we persevere with the maths and note that if

$$\begin{aligned} \sin\theta_T &= \frac{n_2}{n_1}\sin\theta_I > 1 \\ \text{then} \qquad \cos\theta_T &= (1-\sin^2\theta_T)^{1/2} = i\left(\left(\frac{n_2}{n_1}\sin\theta_I\right)^2 - 1\right)^{1/2} \end{aligned}$$

clearly we can't interpret θ_T as an angle any more but the maths is valid

One can show that

$$\underline{E}_{evanescent} = \underline{E}_T e^{i(\underline{k}_T\cdot\underline{r}-\omega t)} = \underline{E}_T e^{i(kx-\omega t)}\, e^{-z/\alpha}$$

where

$$k = \frac{\omega n_1}{c}\sin\theta_I \qquad \alpha^{-1} = \frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_I - n_2^2}$$

with $\alpha \simeq$ the wavelength ($\sim 0.5\mu$m). So we have **attenuation** in the z direction

The transmission coefficient $t \neq 0$ but no energy is carried into medium 2.
Instead there is a travelling wave directed along the interface, which decays in the z direction (into medium 2):

The decay is **not** adsorption **or** the skin effect but can be though of as **Light tunnelling**: light can tunnel across a thin layer of medium 2 via the evanescent wave.