EM 3 Section 4: Poisson's Equation

4. 1. Poisson's Equation

If we replace $\underline{E}$ with $-\underline{\nabla}V$ in the differential form of Gauss's Law we get **Poisson's Equation**:

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \tag{1}$$

where the Laplacian operator reads in Cartesians $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

It relates the second derivatives of the potential to the local charge density.

In a region absent of free charges it reduces to **Laplace's equation**:

$$\boxed{\nabla^2 V = 0} \tag{2}$$

Note that one solution is a uniform potential $V = V_0$, but this would only apply to the case where there are no free charges anywhere. More generally we have to solve Laplace's equation subject to certain *boundary conditions* and this yields non-trivial solutions.

Poisson's and Laplace's equations are among the most important equations in physics, not just EM: fluid mechanics, diffusion, heat flow etc. They can be studied using the techniques you have seen Physical Mathematics e.g. separation of variables, orthogonal polynomials etc

4. 2. Solutions of Poisson's Equation: helpful properties

If you know V everywhere you can find ρ at any point by differentiating twice.

Example

$$V = a + bx^2 y$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 2by + 0 + 0$$

$$\rho(x, y, z) = -\epsilon_0 \nabla^2 V = -2\epsilon_0 by$$

If you know ρ everywhere you can find V at any point but you have to solve Poisson's equation. *This is a harder but much more common task!* It is the central problem of electrostatics.

Helpful Property 1: Linearity

If $\nabla^2 V_1 = -\frac{\rho_1(\underline{r})}{\epsilon_0}$ and $\nabla^2 V_2 = -\frac{\rho_2(\underline{r})}{\epsilon_0}$ then

$$\nabla^2 (V_1 + V_2) = -\frac{\rho_1(\underline{r}) + \rho_2(\underline{r})}{\epsilon_0}$$

This property is known as *linearity* and gives rise to the **"superposition principle"** which is used to sum potentials V arising from different point charges or integrate over the contributions from a charge distribution.

Warning: must check BC's still satisfied

Superposition also applies to $\underline{E} = -\underline{\nabla} V$

often easier to superpose V (scalar) than $\underline{E}$ (vector), but not always: go case by case

Helpful Property 2: Uniqueness Theorem

If a potential obeys Poisson's equation and satisfies the known boundary conditions it is the **only** solution to a problem. This is known as the *uniqueness* theorem.

Basically if one can find a solution by whatever means —usually educated guesswork—then it is the unique solution.

Proof of Uniqueness Consider region $\mathcal{R}$ with boundary $\mathcal{B}$. Let $\rho(\underline{r})$ be specified within $\mathcal{R}$

Figure 1: Diagram of region and boundary for Uniqueness Theorem (c.f. Griffiths Fig 3.4)

BCs: suppose either

(i) V is specified on $\mathcal{B}$

(ii) $\underline{E} = -\underline{\nabla} V$ is specified on $\mathcal{B}$

Then **any** solution of Poisson's equation obeying the BCs is the **only** solution [up to a boring additive constant in case (ii)] **NB**: $\mathcal{B}$ could be at infinity

Proof of Theorem

Suppose there are 2 different solutions, V_1, V_2. Define $\psi = V_1 - V_2$

Then $\nabla^2 \psi = 0$ in $\mathcal{R}$ (Laplace's equation)

BCs on $\mathcal{B}$: case (i) $\psi = 0$; case (ii) $\underline{\nabla} \psi = 0$

Laplace: $\nabla^2 \psi = 0 \qquad \Rightarrow \qquad \psi \nabla^2 \psi = 0$

$$\Rightarrow \qquad \underline{\nabla} \cdot (\psi \underline{\nabla} \psi) - (\underline{\nabla} \psi)^2 = 0$$

$$\Rightarrow \qquad \int_{\mathcal{R}} \left[\underline{\nabla} \cdot (\psi \underline{\nabla} \psi) - (\underline{\nabla} \psi)^2 \right] d\tau = 0$$

Apply divergence theorem to first term

$$\int_{\mathcal{B}} \psi \underline{\nabla} \psi . \underline{\mathrm{d}S} - \int_{\mathcal{R}} (\underline{\nabla} \psi)^2 \, d\tau = 0$$

First term now zero by BCs in either case

The remaining integrand is non-negative, so it must vanish to get zero for the integral

$$\underline{\nabla} \psi = 0 \qquad \Rightarrow \qquad \psi = V_1 - V_2 = \text{const}$$

The constant is zero by BCs in case (i)

4. 3. Simple Example: hollow conductor

Figure 2: Diagram of cavity in a conductor

Consider a cavity $\mathcal{R}$ in a conductor **Claim:** If $\rho = 0$ in cavity, then $\underline{E} = \underline{0}$ inside

Proof: Inner surface is equipotential (since it is conducting), $V = V_0$. This gives our boundary condition

Now inside the cavity $\nabla^2 V = 0$ since there is no charge.

Thus we must solve Laplace's equation subject to the condition that V is constant along the (closed) boundary.

But one solution is $V = V_0$ everywhere within $\mathcal{R}$ and this satisfies boundary condition.

Uniqueness: this is the **only** solution $\Rightarrow \underline{E} = -\underline{\nabla} V = \underline{0}$ for **any** charge-free cavity within a conductor.

4. 4. The Method of Images

This is a technique for guessing (and then verifying) the solution to Poisson's equation. Due to uniqueness it is then the only solution.

The idea is to place a suitable set of "image charges" external to the physical region of the field, in such a way that they generate the required boundary conditions, without affecting Poisson's equation within the physical region (since an image charge is not in the physical region).

Point charge near a conducting plane

Consider a point charge, Q, a distance a from a flat conducting surface at a potential $V_0 = 0$.

Figure 3: Point charge near a conducting plane

The problem is to solve Poisson's equation with a point charge at $a\underline{e}_z$ and boundary condition that $V = 0$ on the boundary ($z = 0$) of the physical region $z \geq 0$.

Now the potential from the point charge at $a\underline{e}_z$ is

$$V = \frac{1}{4\pi\epsilon_0} \frac{1}{(x^2 + y^2 + (z-a)^2)^{1/2}} \tag{3}$$

The idea is to consider an 'image charge' in the unphysical region $z < 0$. In the physical region ($z > 0$) the potential due to such an image charge satisfies Laplace's equation therefore we can simple add it to (3) and still satisfy Poisson's equation.

The correct guess for the image charge is $-q$ at $-a\underline{e}_z$. This basically reflects the symmetry of the problem. To check that the boundary condition is actually satisfied we write out the potential

$$V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{(x^2 + y^2 + (z-a)^2)^{1/2}} - \frac{1}{(x^2 + y^2 + (z+a)^2)^{1/2}} \right] \tag{4}$$

and see that it vanishes at $z = 0$ as required.

N.B. if the conducting sheet is at potential $V_0 \neq 0$ we simply add a constant V_0 to (4).

$\underline{E}$ must be normal to the conducting surface $\underline{E} = E_z \underline{e}_z$. Therefore on the surface

$$\begin{aligned} E_z &= -\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{-q}{4\pi\epsilon_0} \left[-\frac{(z-a)}{(x^2 + y^2 + (z-a)^2)^{3/2}} + \frac{(z+a)}{(x^2 + y^2 + (z+a)^2)^{3/2}} \right] \Bigg|_{z=0} \\ &= \frac{-q}{2\pi\epsilon_0} \frac{a}{(x^2 + y^2 + a^2)^{3/2}} \end{aligned}$$

and the surface charge density is

$$\sigma = \frac{-q}{2\pi} \frac{a}{(x^2 + y^2 + a^2)^{3/2}}$$

Integrating this over the whole surface (left as exercise) shows that the surface charge is $-q$. *Note that there is an attractive force between the charge distribution on the conducting surface and the point charge above it.*

The method of images is really an inspired guess which works when the problem has appropriate symmetry (see tutorial problems for further examples).