EM 3 Section 5: Electric Dipoles

An electric dipole is formed by two point charges $+q$ and $-q$ connected by a vector $\underline{a}$. The **electric dipole moment** is defined as $\boxed{\underline{p} = q\underline{a}}$.

By convention the vector $\underline{a}$ points from the negative to the positive charge. Here we also take the origin to be at the centre and $\underline{a}$ to be aligned to the z axis (see diagram)

Figure 1: Diagram of electric dipole aligned along z axis. Griffiths Fig 3.37

5. 1. Field of an electric dipole

We first calculate the potential and then the field:

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{1}$$

where $r_\pm$ are the distances from the +ve(-ve) charge to the point $\underline{r}$.

Now

$$r_\mp^2 = |\underline{r} \pm \underline{a}/2|^2 = (r^2 \pm \underline{a}\cdot\underline{r} + a^2/4) = r^2\left(1 \pm \frac{a}{r}\cos\theta + \frac{a^2}{4r^2}\right)$$

Now consider the "far field limit" $\boxed{r \gg a}$

$$\begin{aligned}\frac{1}{r_\pm} &= \frac{1}{r}\left(1 \mp \frac{a}{r}\cos\theta + \frac{a^2}{4r^2}\right)^{-1/2} \\ &\simeq \frac{1}{r}\left(1 \pm \frac{a}{2r}\cos\theta + O((a/r)^2)\right)\end{aligned}$$

where $O((a/r)^2)$ indicates terms proportional to $(a/r)^2$ or higher powers

Thus we obtain in the far-field limit

$$V = \frac{qa\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\underline{p}\cdot\hat{\underline{r}}}{4\pi\epsilon_0 r^2} \tag{2}$$

One can check that away from the charges this is a solution of Laplace's equation (see tut)

The components of the electric field $\underline{E} = -\underline{\nabla}\mathrm{V}$ are simplest in spherical polar coordinates:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \qquad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \tag{3}$$

To get a co-ordinate free form of the electric field we can use (see tutorial sheet 1)

$$\underline{E} = -\underline{\nabla}V = -\frac{1}{4\pi\epsilon_0}\underline{\nabla}\left(\frac{\underline{p}\cdot\hat{\underline{r}}}{r^2}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\underline{p}\cdot\underline{r})\underline{r}}{r^5} - \frac{\underline{p}}{r^3}\right) \tag{4}$$

The important point to note is that a dipole field is $1/r^3$*, whereas a point charge field is* $1/r^2$

N.B. The above 'far-field' limit $r \gg a$ can also be presented as an *ideal dipole* which is the limit $\underline{a} \to 0$, $q \to \infty$ but $\underline{p}$ finite. The ideal dipole is a useful approximation to the 'physical dipole' for which the potential is given by (1) and

$$\underline{E} = -\underline{\nabla}V = \frac{q}{4\pi\epsilon_0}\left[\frac{\underline{r} - \underline{a}/2}{|\underline{r} - \underline{a}/2|^3} - \frac{\underline{r} + \underline{a}/2}{|\underline{r} + \underline{a}/2|^3}\right] \tag{5}$$

However the sketches look a little different:

Figure 2: Sketch of electric dipole field: ideal and 'physical' *Griffiths Fig 3.37*

Why dipoles matter I: Many molecules have a **permanent** dipole moment $\underline{p}$ (e.g. H_20)
All others, and all **atoms**, acquire an **induced** dipole when placed in $\underline{E}$ field
Since atoms and molecules are (a) neutral and (b) almost pointlike, the ideal dipole concept is crucial to understanding media see later.

5. 2. Interaction of dipole with external Electric Field

To calculate the force on a dipole in an external field $\underline{E}_{\rm ext} = -\underline{\nabla}\phi$ (note here we use ϕ for the electrostatic potential of the external field) it is simplest to first calculate the potential energy of the dipole in this field:

$$\begin{aligned}U_{\rm dip} &= q\phi(\underline{a}/2) - q\phi(-\underline{a}/2) \\ &\simeq q\underline{a}\cdot\underline{\nabla}\phi = -\underline{p}\cdot\underline{E}_{\rm ext}\end{aligned}$$

where we have made a Taylor expansion to first order in $\underline{a}$.

Thus $U_{\rm dip}$ is minimised when the dipole is parallel to the field and maximised when antiparallel: you need $\Delta U = 2p|E_{\rm ext}|$ to reverse the direction.

Moreover we can compute the force felt by the dipole through

$$\underline{F} = -\underline{\nabla} U_{\rm dip} = \underline{\nabla}(\underline{p}\cdot\underline{E}_{\rm ext}) \tag{6}$$

There is *no net force* on a dipole in a *uniform* electric field ($\underline{E}_{\rm ext}$ does not depend on $\underline{r}$), since the two charges of a dipole experience equal and opposite force. However, in a *non-uniform* field there is a force which moves the dipoles along the field gradient.

In a *uniform* external field there is still a *torque* which acts to align the dipole moment $\underline{p}$ along the direction of $\underline{E}_{\rm ext}$:

$$\underline{T} = (\underline{a}/2)\times(q\underline{E}_{\rm ext}) - (\underline{a}/2)\times(-q\underline{E}_{\rm ext}) = \underline{p}\times\underline{E}_{\rm ext} \tag{7}$$

The work done by the torque in rotating the dipole from an aligned position through an angle θ relative to the field is:

$$W = \int_0^\theta T d\theta = \int_0^\theta pE_{\rm ext}\sin\theta d\theta = pE_{\rm ext}(1-\cos\theta) \tag{8}$$

To summarise

i) dipoles tend to align with an external field

ii) dipoles migrate up field gradients

5. 3. Why dipoles matter II

Let's go back to the Taylor expansion of the electrostatic potential, this time for an *arbitrary* bounded charge distribution confined to some region $\mathcal{R}$

$$\begin{aligned}
V(\underline{r}) &= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{R}} {\rm d}V' \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|} \\
&= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{R}} {\rm d}V' \frac{\rho(\underline{r}')}{(r^2 - 2\underline{r}\cdot\underline{r}' + r'^2)^{1/2}} \\
&= \frac{1}{4\pi\epsilon_0}\int_{\mathcal{R}} {\rm d}V' \frac{\rho(\underline{r}')}{r}\left(1 - \frac{2\underline{r}\cdot\underline{r}'}{r^2} + \frac{(r')^2}{r^2}\right)^{-1/2} \\
&\simeq \frac{1}{4\pi\epsilon_0}\int_{\mathcal{R}} {\rm d}V' \rho(\underline{r}')\left[\frac{1}{r} + \frac{\hat{\underline{r}}\cdot\underline{r}'}{r^2} + \frac{3(\hat{\underline{r}}\cdot\underline{r}')^2 - (r')^2}{2r^3} + \cdots\right]
\end{aligned}$$

In the last line we have used the Taylor expansion

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

and gathered together terms according to the power of r i.e. we have made a far-field expansion ($r \gg 1$) in powers of $1/r$.

We can tidy up the expansion to write it as

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + \frac{1}{4\pi\epsilon_0}\frac{\hat{\underline{r}}\cdot\underline{P}}{r^2} + \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\frac{1}{2}\sum_{i,j}\mathcal{Q}_{ij}\hat{r}_i\hat{r}_j + \cdots \tag{9}$$

where

$$\begin{aligned}
Q &= \int_V {\rm d}V'\rho(\underline{r}') \\
\underline{P} &= \int_V {\rm d}V'\underline{r}'\rho(\underline{r}') \\
\mathcal{Q}_{ij} &= \int_V {\rm d}V'(3r'_i r'_j - (r')^2\delta_{ij})\rho(\underline{r}')
\end{aligned}$$

Thus Q is the total net charge ; $\underline{P}$ is the dipole moment and $\frac{\hat{\underline{r}}^T\mathcal{Q}\hat{\underline{r}}}{r^3}$ is the quadupole moment with $\mathcal{Q}$ the quadrupole tensor (a basis dependent matrix). Each of the monopole, dipole, quadrupole fields have distinct and characteristic shapes.

The first term ('monopole term') would normally be the dominant term and this would quite reasonably represent the far-field charge distribution as that of a single lump of charge at the origin.

However, when the total charge Q vanishes (as in the case of a dipole) it is the next term ('dipole term') which is dominant. One can think of charge distributions where the second term term also vanishes (e.g. electric quadrupole where $\underline{P} = 0$ see figure) then the dominant term becomes the third term (quadrupole term) and so on.

Figure 3: 4 charges forming an electric quadrupole *Griffiths Fig 3.27*

Significantly, each term in the expansion (9) is separately a solution of Laplace's equation (outside of the region V that contains the charges).

Thus one can use the idea of the method of images and use an image dipole

$$V = \frac{1}{4\pi\epsilon_0}\frac{(\widehat{\underline{r}-\underline{r}'})\cdot\underline{P}}{|\underline{r}-\underline{r}'|^2}$$

at some point $\underline{r}'$ *outside* of the physical region to solve Poison's equation *inside* the physical region, and fix certain boundary conditions on the boundary of the physical region.

An example is a conducting sphere in a uniform external field $\underline{E}_0$. The boundary condition is that $\underline{E}$ is radial at the surface of the sphere which is an equipotential. This is achieved by using an image electric dipole at the centre of the sphere - see tutorial. The result is that on the surface of the sphere there is an induced charge distribution which is positive on one side and negative on the other.