EM 3 Section 7: Magnetic force, Currents and Biot Savart Law

7. 1. Magnetic force

The magnetic field $\underline{B}$ is defined by the force on a moving charge:

$$\boxed{\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})} \quad (1)$$

This is the **Lorentz Force Law**. The second term is the magnetic force. The unit of magnetic field is the Tesla (T) which is $\mathrm{NA^{-1}m^{-1}}$. Actually this is a pretty big unit and a Gauss $= 10^{-4}$T is more commonly used.

7. 2. Current density and current elements

The first thing to note is that a moving charge by itself does not really constitute a current. Instead we need a moving density of charges. For the moment we will consider **steady** currents so that at any point we have a constant density of charged particles moving past the point (clearly we will need sources of current somewhere but let's not worry about that for the moment). Also we can have zero net charge but a steady current, if the densities of positive and negative particles are the same but their velocities are different. A current consists of n charges q per unit volume moving with average velocity $\underline{v}$. These charges form a local current density:

$$\underline{J} = nq\underline{v} \quad (2)$$

The total current I passing *through* a surface is obtained by integration:

$$\mathrm{I} = \int_A \underline{J} \cdot \mathrm{d}\underline{S} \quad (3)$$

where as usual $\mathrm{d}\underline{S}$ points normal to the surface.

Units

The unit of current is the Ampere (A), which is a base SI unit, 1A = $1\mathrm{Cs^{-1}}$.The unit of bulk current density $\underline{J}$ is $\mathrm{A/m^2}$. We can also have surface current densities usually denoted $\underline{K}$ or $\underline{j}$ (units A/m) and line current densities usually denoted $\underline{I}$ (units A). Calling all of these 'densities' is a bit confusing since none has units of current per unit volume but that is the way it is!

What we shall see is that steady currents play the key role in magnetism as do electric charges in electrostatics, that is

Stationary charges	$\Rightarrow$	constant electric fields: electrostatics
Steady currents	$\Rightarrow$	constant magnetic fields: magnetostatics

A **current element**, denoted here $\mathrm{d}\underline{\mathcal{I}}$, has units Am and is a vector

$$\begin{array}{rcll} \mathrm{d}\underline{\mathcal{I}}(\underline{r}) & = & \underline{J}(\underline{r})\mathrm{d}V & \text{Current element in bulk} \\ \mathrm{d}\underline{\mathcal{I}}(\underline{r}) & = & \underline{K}(\underline{r})\mathrm{d}S & \text{Current element on surface} \\ \mathrm{d}\underline{\mathcal{I}}(\underline{r}) & = & \underline{I}(\underline{r})\mathrm{d}l & \text{Current element along a wire} \end{array}$$

Warning: you need to take care with current elements e.g. $\underline{K}\mathrm{d}S \neq K\mathrm{d}\underline{S}$ since the left hand side points in the direction of the current vector on the surface, but the right hand side points normal to the surface. On the other hand $\underline{I}\mathrm{d}l = I\mathrm{d}\underline{l}$ since a line current element always points along the wire in the direction $\mathrm{d}\underline{l}$.

Figure 1: Diagram of rotating charged disc

Example: Rotating disc An insulating disc of radius R, carrying a uniform surface charge density σ, is mechanically rotated about its axis with an angular velocity vector $\underline{\omega}$ (in the $\underline{e}_z$ direction). As a result it has a current density on its surface:

$$\underline{K} = \sigma\underline{v} = \sigma\underline{\omega} \times \underline{r} = \sigma r\omega\underline{e}_\phi \tag{4}$$

Note that the current density increases linearly with r. Check that you understand how the direction comes from the right hand rule.

Conductivity

The quantity σ (conductivity) describes the intrinsic conduction properties of a bulk material in response to an electric field. The current density is:

$$\underline{J} = \sigma\underline{E} \tag{5}$$

Note that σ is a property of a particular material, and that it depends on temperature. A typical value of σ for a metal is $6 \times 10^7\ \Omega^{-1}m^{-1}$.

Sometimes resistivity defined as $\rho = \dfrac{1}{\sigma}$ is used.

Remark: Actually (5) makes a crucial assumption that $\underline{J}$ and $\underline{E}$ are parallel which is not necessarily the case for some non-isotropic materials where, for example, current can only flow in certain directions. In such cases one needs a conductivity tensor.

The force on a steady current element is

$$\boxed{\mathrm{d}\underline{F} = \mathrm{d}\underline{\mathcal{I}} \times \underline{B}} \tag{6}$$

In particular, if the current element comes from a bulk current density $\underline{d\mathcal{I}} = \underline{J}\,dV$ we have:

$$\underline{dF} = \underline{J} \times \underline{B}\,dV \tag{7}$$

7. 3. Biot Savart Law and Calculation of Magnetic Fields

Just as a charge creates an electric field, so a current element at $\underline{r}'$ creates a magnetic field at a position $\underline{r}$:

$$\underline{dB}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{\underline{d\mathcal{I}}(\underline{r}') \times (\widehat{\underline{r} - \underline{r}'})}{|\underline{r} - \underline{r}'|^2} \tag{8}$$

This is known as the **Biot-Savart Law**. The law was established expermentally; we shall take it as our starting point. It plays the same role for magnetostatics as Coulomb's law for the electric field due to a point charge in electrostatics.

The constant μ_0 is known as the *permeability* of free space:

$$\mu_0 = 4\pi \times 10^{-7}\text{Hm}^{-1} \tag{9}$$

The direction of the field is perpendicular to $\underline{d\mathcal{I}}(\underline{r}')$ and $\widehat{\underline{r} - \underline{r}'}$, the vector from the current element to the point $\underline{r}$. The direction can be remembered from the usual right hand rule for vector products.

Note that **superposition** holds for magnetic fields, therefore the magnetic field can be calculated by integration over current elements:

$$\boxed{\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int\frac{\underline{d\mathcal{I}}(\underline{r}') \times (\widehat{\underline{r} - \underline{r}'})}{|\underline{r} - \underline{r}'|^2}} \tag{10}$$

e.g. for integration over a volume containing a distribution of current density:

$$\boxed{\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int_V\frac{\underline{J}(\underline{r}') \times (\widehat{\underline{r} - \underline{r}'})}{|\underline{r} - \underline{r}'|^2}dV'} \tag{11}$$

7. 4. Magnetic Force between Currents

Substituting the Biot Savart Law for the magnetic field due to a current element into (6)

$$\boxed{\underline{dF}_{12} = \frac{\mu_0}{4\pi r_{12}^2}\underline{d\mathcal{I}}_1 \times (\underline{d\mathcal{I}}_2 \times \underline{\hat{r}}_{12})} \tag{12}$$

which corresponds to the **force between a pair of current elements**. Here $\underline{\hat{r}}_{12}$ points from $\underline{d\mathcal{I}}_1$ to $\underline{d\mathcal{I}}_2$ Equation (12) is also referred to as Biot-Savart law and is the equivalent of Coulomb's law for the force between two point charges.

If the current elements are due to current densities we have

$$\underline{dF}_{12} = \frac{\mu_0}{4\pi r^2}\underline{J}_1 \times (\underline{J}_2 \times \underline{\hat{r}})dV_1dV_2 \tag{13}$$

where the force is attractive for parallel currents, and repulsive for antiparallel currents.

Figure 2: Diagram for calculating $\underline{B}$ from an infinite straight wire *Griffiths Fig. 5.18*

7. 5. Example of long straight wire

We consider a long straight wire which we choose to be along the z axis so that a point $\underline{r}'$ on the wire is given by $\underline{r}' = z'\underline{e}_z$. We want to compute the field using (10). It is best to use cylindrical polars: we choose the origin along the wire so that $\underline{r}$ is $\perp$ to $\underline{e}_z$ i.e. $\underline{r} = \rho\underline{e}_\rho$ where ρ is the radial distance of the point from the wire.

Now $\underline{\mathrm{d}\mathcal{I}} = I\mathrm{d}z'\underline{e}_z$ and $\underline{\mathrm{d}r'} = \mathrm{d}z'\underline{e}_z$ so

$$\underline{\mathrm{d}r'} \times (\underline{r} - \underline{r}') = \underline{\mathrm{d}r'} \times \underline{r} = \rho\mathrm{d}z'\underline{e}_\phi$$

and we find

$$\underline{B}(\underline{r}) = \frac{\mu_0 I\rho}{4\pi}\underline{e}_\phi \int_{-\infty}^{\infty} \frac{\mathrm{d}z'}{|\rho^2 + z'^2|^{3/2}}$$

To evaluate the remaining integral we use substitution $z' = \rho\tan\theta$ so that $\mathrm{d}z' = \rho\sec^2\theta\,\mathrm{d}\theta$ and $\rho^2 + z'^2 = \rho^2\sec^2\theta$. Then we obtain

$$\underline{B}(\underline{r}) = \frac{\mu_0 I}{4\pi\rho}\underline{e}_\phi \int_{-\pi/2}^{\pi/2} \cos\theta\mathrm{d}\theta = \frac{\mu_0 I}{2\pi\rho}\underline{e}_\phi$$

We now consider the force on a current element of a second parallel wire (at distance d

Figure 3: Two parallel wires separated by distance d (*Griffiths Fig. 5.20*)

from the first) coming from the magnetic field $\underline{B}_1(\underline{r})$ due to the first wire. Again we choose coordinates so that this current element lies at $\underline{r} = \rho\underline{e}_\rho$ in cylindrical polars

$$\begin{aligned}\underline{\mathrm{d}F} &= \underline{\mathrm{d}I}_2(\underline{r}) \times \underline{B}_1(\underline{r}) = I_2\mathrm{d}z\underline{e}_z \times \frac{\mu_0}{2\pi d}I_1\underline{e}_\phi \\ &= -\frac{\mu_0 I_1 I_2}{2\pi d}\mathrm{d}z\underline{e}_\rho\end{aligned}$$

There is an *attractive* force per unit length between two parallel infinitely long straight wires: this is the basis of the definition of the Ampère.