Junior Honours

Electromagnetism **Problem Sheet 4**

Electrostatics: Electric field Energy; Magnetostatics: Biot-Savart

The questions that follow on this and succeeding sheets are an integral part of this course. Cross references to the questions are given in the lecture notes. The code beside each question has the following significance:

- **K**: key question – explores core material
- **R**: review question – an invitation to consolidate
- **C**: challenge question – going beyond the basic framework of the course
- **S**: standard question – general fitness training!

4.1 **Where do you get the energy? [K]**

A conducting sphere of radius a carries a charge Q on its surface.

(i) Find its potential $\varphi(r)$ for $r > a$ (choosing $\varphi = 0$ very far away), and thus the potential $V(Q) = \varphi(a)$ on the surface of the sphere.

(ii) Explain why the electrostatic energy of the charged sphere can be written $W_e = \int_0^Q V(Q')\mathrm{d}Q'$, and hence calculate it.

(iii) Show that this gives the same answer as found by the formula $W_e = \frac{\epsilon_0}{2} \int |\underline{E}(\underline{r})|^2 \,\mathrm{d}V$ for the total energy in the electric field, where the volume integral is over all space.

4.2 **Parallel spheres? [S]**

Two thin concentric, conducting spherical shells have radii a and $b > a$. They carry charges $+Q$ and $-Q$ respectively.

(i) Use Gauss's law and symmetry to find the field $\underline{E}$ at a radius r from the centre. Sketch the field lines.

(ii) Find the potential difference between the shells, and hence show that the capacitance is $C = \frac{4\pi\epsilon_0 ab}{b-a}$.

(iii) Show that this reduces to the form for a parallel plate capacitor of the appropriate area, in the limit where b is only slightly bigger than a.

4.3 **Atomic nucleus [S]** (a) Assuming that the electric charge Ze of an atomic nucleus is uniformly distributed inside a sphere of radius R, obtain the electric field $\underline{E}$ and electric potential φ both inside and outside.

(b) Obtain an expression for the electrostatic energy $\frac{1}{2} \int \rho \, \varphi \,\mathrm{d}V$ of the nucleus and verify that it is equal to the field energy $\frac{1}{2}\epsilon_0 \int |\underline{E}|^2 \,\mathrm{d}V$ (where the integral is over all space).

4.4 **Under pressure [S]**

A cylindrical pipe of radius 1cm and length 1m is filled with mercury (conductivity $\sigma = 1.1 \times 10^6 \Omega^{-1}\mathrm{m}^{-1}$). A potential difference 100 V acts across the two ends of the pipe, creating an electric current through the mercury (which remains stationary).

(i) Find the current density, assumed uniform, within the mercury. What is the total current $I_{enc}(r)$ enclosed by a circle of radius r, coaxial with the cylinder? Using Ampère's law, find the $\underline{B}$ field at radius r.

(ii) Consider a small volume dV of the mercury at radial distance r. Show that the $\underline{B}$ field, acting on the current element $\underline{J}$dV, produces a radial force $d\underline{F} = f\hat{\underline{r}}\mathrm{d}V$. Find the sign and magnitude of the local force per unit volume f.

(iii) In practice this is balanced by a radial pressure gradient: $f_r = \partial p/\partial r$. Find the pressure difference Δp between the centre and the edge of the pipe.

[Remark – exactly the same force is present in a solid wire, but the balancing elastic force is more complicated than a simple pressure field.]

4.5 **A regrettable navigational error [S]**

The Earth's magnetic field arises from circulating currents in its core, and can be approximated as that of a dipole at the centre. Assume for this question that the axis of the Earth's dipole coincides with the axis of rotation, although they are actually several degrees apart. On the equator (6400 km from the centre), the field strength is $4 \times 10^{-5}T$, with $\underline{B}$ directed horizontally, pointing due North.

(i) Find the dipole moment. If this is modelled as arising from a simple current loop whose radius is 0.1 times that of the Earth, what is the current?

(ii) Find the field strength at the geographical place called "the North Pole". Which way does the $\underline{B}$ field point there? (A sketch of the field lines may help.) Deduce that "the North Pole" is actually the south pole of the Earth, viewed as a bar magnet.

4.6 **In a spin [R/S]** (first part done in lectures)

(i) Show that the $\underline{B}$ field on the axis of a circular current loop of radius a is

$$\underline{B} = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \underline{e}_z$$

with z the distance along the axis from the centre of the loop.

(ii) An insulating disc of radius s has uniform surface charge density σ. It rotates at angular velocity ω about a perpendicular axis through its centre. What is the surface current density $\underline{j}(\underline{r})$ at position $\underline{r}$ relative to its centre?

(iii) Consider that part of the disc with radii between r and $r + \delta r$. Find its contribution $\delta\underline{B}(z)$ to the field on the axis.

(iv) Hence give as an integral the field on the axis for the spinning disc, and evaluate this at the disc's centre.

(v) Show that as $z \to \infty$,

$$\underline{B}(z) \sim \frac{1}{8}\mu_0 \sigma \omega \frac{a^4}{z^3} \underline{e}_z.$$

(vi) What would the corresponding results be for a spinning ring of inner radius a and outer radius b? Recover the result in part (i) by taking the limit $b \to a$.