STATISTICAL PHYSICS 06/07

The Microcanonical, Canonical and Grand Canonical Distrutions **Tutorial Sheet 2**

The questions that follow on this and succeeding sheets are an integral part of this course. The code beside each question has the following significance:

- **K**: key question – explores core material
- **R**: review question – an invitation to consolidate
- **C**: challenge question – going beyond the basic framework of the course
- **S**: standard question – general fitness training!

2.1 **Revision of Model Magnet** [r] When a particle with spin $^1/_2$ is placed in a magnetic field H, its energy level is split into $\pm\mu H$ and it has a magnetic moment μ or $-\mu$ along the direction of the magnetic field, respectively. Suppose that an assembly of N such particles on a lattice is placed in a magnetic field H and is kept at temperature T. Find the internal energy, the entropy, the heat capacity and the total magnetic moment M of this assembly.

Sketch the internal energy and heat capacity, identifying and analyisng the low temperature and high temperature regimes.

2.2 **Magnetisation Fluctuations in General Magnetic System** [s] An assembly of spin 1/2 at a fixed temperature T is subject to an applied magnetic field H. If the net magnetic moment of the assembly in state i is M_i, in the direction of the field, then the associated total energy is given by

$$E_i = E_i(H=0) - \mu_0 M_i H\ ,$$

where $E_i(H=0)$ represents the mutual interaction of the individual spins in the absence of an external field, and μ_0 is the permeability of free space.

Show that fluctuations of the magnetic moment about its mean value are given by

$$\Delta M^2 \equiv \langle M^2\rangle - \langle M\rangle^2 = \frac{kT\chi}{\mu_0}$$

where χ is the isothermal magnetic susceptibility defined by

$$\chi = [\partial\langle M\rangle/\partial H]_T\ .$$

2.3 **Quantum rotor: classical and quantum limits** [s]

$$\epsilon_r = \frac{\hbar^2}{2I} r(r+1) \quad , \quad r = 0, 1, 2 ...$$

where I is a constant and the level ϵ_r is $(2r+1)$-fold degenerate.
(a) Write down the canonical partition function of the rotational motion, and obtain manageable approximations for it (i) at low temperatures and (ii) at high ones.
[Hint: to decide on suitable approximations, consider in each limit how many terms matter in the sum. If only one or two terms matter, neglect the rest. If many matter, consider how a sum of many terms (if the summand is slowly varying) can be approximated by an integral.]
(b) Find the mean energy $E(T)$ in each limit and hence obtain expressions for the heat capacity at low and high temperatures.
[Answers: $c_{V,rot} = 3\hbar^4 I^{-2} k_B^{-1} T^{-2} \exp[-\beta\hbar^2/I] \simeq 0$ for $k_B T \ll \epsilon_1$; $c_{V,rot} = k_B$ for large T.]

2.4 **Stirling's formula and entropy [r]**

a) Starting from Stirling's formula

$$\ln(N!) \simeq N[\ln N - 1]$$

show using Boltzmann's formula for entropy that, for a lattice of $\mathcal{N}$ independent spins, each either up or down, the entropy is given by

$$S = -k\mathcal{N}[c \ln c + (1-c)\ln(1-c)]$$

where c is the fraction of up spins

Redrive this result from the Gibbs entropy

(b) From Stirling's formula, show that if, instead of having spins, each lattice site can be either empty, or occupied by a particle of type 1 (concentration c_1) or occupied by a particle of type 2 (concentration c_2), then

$$S = -k\mathcal{N}[c_1 \ln c_1 + c_2 \ln c_2 + (1-c_1-c_2)\ln(1-c_1-c_2)]$$

Rederive this result using the Gibbs entropy.

2.5 **Grand Potential [r]** Prove from the grand partition function that the grand potential is

$$\Phi = \langle E \rangle - TS_{gibbs} - \mu \langle N \rangle$$

(Hint: Start from the expression for S_{gibbs}.)

Show also that for a single species 'PVT' systen

$$\Phi = -PV$$

2.6 **Equivalence of ensembles [k]** It is sometimes said that, in the limit of large N, the canonical ensemble becomes equivalent to the microcanonical ensemble. How can this statement be justified? [Hint consider energy fluctuations] Similar when does the grand canonical ensemble become equivalent to the canonical ensemble? When might we expect equivalences to break down?

2.7 **Correlation of energy and pressure fluctuations [s]** Show that the fluctuations in the pressure and total energy of a fluid in a container of fixed volume V, satisfy the relation

$$\langle \Delta E \, \Delta P \rangle = kT^2 \left(\frac{\partial \langle P \rangle}{\partial T} \right)_V ,$$

provided that the fluid is in thermal equilibrium.