Mesoscopics and Quantum Transport – Solutions to Problems

1. Note that if we swap land and water in a system with $p\%$ land then we obtain one with $(100-p)\%$ land. This symmetry implies one of 2 consequences:

- there is no coexistence of paths accross the whole system on land and water; the boundary must be at half–filling, 50%.
- paths on water and land can coexist.

The system is supposed to be isotropic. If coexistence is allowed then it should be possible to have continuous land from left to right and continuous water from top to bottom. This is clearly impossible in 2D (but not in 3D). Hence the transition must be at half filling.

2. From the information given in the notes we have a smallest energy splitting $\Delta E = [\rho \frac{4}{3}\pi R^3]$ and

$$P(R) \sim \exp\left(-\alpha R - \beta/[\rho \tfrac{4}{3}\pi R^3]\right)$$

Find the optimum hop length by minimising the argument of the exponential

$$\begin{aligned}
0 &= \alpha - 9\beta/[4\pi\rho R^4] \\
\Rightarrow \quad R &= \left(\frac{9\beta}{4\pi\rho\alpha}\right)^{1/4} \\
\Rightarrow \quad P(R) &\sim \exp\left[-\left(\frac{9\alpha^3}{4\pi\rho k_\mathrm{B}T}\right)^{1/4} - \left(\frac{3}{4\pi\rho k_\mathrm{B}T}\right)\left(\frac{9}{4\pi\rho\alpha k_\mathrm{B}T}\right)^{-3/4}\right] \\
\Rightarrow \quad \sigma &\sim \exp\left[-\frac{4}{3}\left(\frac{9\alpha^3}{4\pi\rho k_\mathrm{B}T}\right)^{1/4}\right] \\
\Rightarrow \quad T_0 &= \frac{64}{9\pi}\frac{\alpha^3}{\rho k_\mathrm{B}}
\end{aligned}$$

In general the numerical coefficient is unimportant, but rather the functional dependence on α and ρ.

3. Replace equation 8 with $\epsilon = \alpha R$, the solution of the 1D Laplace's equation. Eliminate R with equation 7 to obtain

$$\begin{aligned}
\alpha\epsilon &= \frac{1}{\epsilon\rho} \\
\Rightarrow \quad \rho &\sim \epsilon^{-2} \\
&\sim |E - E_\mathrm{F}|^{-2}
\end{aligned}$$

a singularity.

4. Replace equation 2 with

$$\begin{aligned}
\delta E &\sim 1/\rho R^d \\
&\mapsto 1/\Delta E^\delta R^d \\
\Rightarrow \quad \Delta E &\sim R^{-d/(\delta+1)}
\end{aligned}$$

Now minimise the exponent with respect to R

$$\begin{aligned} 0 &= \alpha - \beta \left(\frac{d}{\delta+1}\right) R^{(d+\delta+1)/(\delta+1)} \\ \Rightarrow \quad R &\sim \left(\frac{\beta}{\alpha}\right)^{(\delta+1)/(d+\delta+1)} \\ \Rightarrow \quad \sigma &\sim \exp\left[-\left(\frac{T_0}{T}\right)^{(\delta+1)/(d+\delta+1)}\right] \end{aligned}$$

The Efros & Shklovskii density of states has $\delta = d-1$ which implies the exponent

$$\frac{\delta+1}{d+\delta+1} = \frac{1}{2}$$

which is independent of d. We therefere expect hopping to give

$$\sigma \sim \exp\left[-(T_0/T)^{1/2}\right]$$

in the presence of Coulomb interactions.

5. We have

$$\begin{aligned} \frac{\mathrm{d}\beta}{\mathrm{d}\ln g} &= \frac{\mathrm{d}}{\mathrm{d}\ln g}\left(1 - \frac{a}{g^n}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}\ln g}\left(1 - a\exp(-n\ln g)\right) \\ &= an\exp(-n\ln g) \\ &= n\frac{a}{g^n} \end{aligned}$$

Now evaluate this when $\beta = 0$ to obtain $\beta' = n$. From equation 25 we conclude that the critical exponent is $1/n$.

6. The crystal obeys Bloch's theorem

$$\psi(r) = \exp(\mathrm{i}kr)U(r)$$

where $U(r)$ is periodic. The kinetic energy may be evaluated

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} &= \frac{\hbar^2}{2m}\left(k^2U - 2\mathrm{i}k\frac{\mathrm{d}U}{\mathrm{d}r} - \frac{\mathrm{d}^2U}{\mathrm{d}r^2}\right)\exp(\mathrm{i}kr) \\ &= \exp(\mathrm{i}kr)\frac{1}{2m}\left(\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}r} + \hbar k\right)^2 U(r) \end{aligned}$$

Compare this with the ring which is threaded by a flux Φ, represented by a vector potential $\boldsymbol{A} = (\Phi/2\pi R)\hat{\theta}$ where R is the radius of the ring. Thus the kinetic energy on the ring may be written

$$\frac{1}{2m}\left(\frac{\hbar}{\mathrm{i}}\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}\theta} - e\frac{\Phi}{2\pi R}\right)\psi(\theta)$$

Comparing these 2 equations gives

$$k = \frac{e\Phi}{2\pi\hbar R}$$

Note that the energy should be periodic in k with period $2\pi/2\pi R = 1/R$ which implies that it should be periodic in Φ with period h/e, a flux quantum for single electrons (compare $h/2e$ for superconductors).

The group velocity for a particular k state in the crystal is given by

$$v_g = \frac{1}{\hbar}\frac{\mathrm{d}E}{\mathrm{d}k}$$

which would imply for the ring

$$v_g = \frac{2\pi R}{e}\frac{\mathrm{d}E}{\mathrm{d}\Phi}$$

Alternatively consider Hamilton's equation $v = \mathrm{d}H/\mathrm{d}p$.

Hence the ring carries a current for almost any finite flux.

This might be detected by measuring the magnetic response of the system, using (e.g.) a SQUID.

Note that the group velocity should scale as the band width, in this case the Thouless energy, and should therefore be small for disordered wires and larger for clean systems such as those formed by using a ring shaped gate on a semiconductor. In addition the contributions from different bands can have different signs, also tending to reduce the signal. This is **not** observed in experiments: the magnetic response of dirty systems is typically much larger than expected and more like that expected from clean systems.

7. We start from

$$\begin{aligned}\sum_i T_i \rho_i \left(\mu_1 - \mu_{\mathrm{B}}\right) &= \sum_i \left(2 - T_i\right)\rho_i \left(\mu_{\mathrm{B}} - \mu_2\right) \\ \sum_i \left(1 + R_i\right)\rho_i \left(\mu_1 - \mu_{\mathrm{A}}\right) &= \sum_i \left(1 - R_i\right)\rho_i \left(\mu_{\mathrm{A}} - \mu_2\right) \quad ,\end{aligned}$$

subtract the 2 equations and then sort the terms containing μ_1 & μ_2 on the left and those containing μ_{A} & μ_{B} on the right.

$$\begin{aligned}&\left[\sum_i T_i\rho_i - \left(1 + R_i\right)\rho_i\right]\mu_1 + \left[\sum_i \left(2 - T_i\right)\rho_i - \left(1 - R_i\right)\rho_i\right]\mu_2 \\ &= \left[\sum_i \left(2 - T_i\right)\rho_i + T_i\rho_i\right]\mu_{\mathrm{B}} - \left[\sum_i \left(1 - R_i\right)\rho_i + \left(1 + R_i\right)\rho_i\right]\mu_{\mathrm{A}}\end{aligned}$$

which leads us to

$$\begin{aligned}&\left[\sum_i \left(-1 + T_i - R_i\right)\rho_i\right]\mu_1 + \left[\sum_i \left(1 - T_i + R_i\right)\rho_i\right]\mu_2 \\ &= \left[\sum_i 2\rho_i\right]\mu_{\mathrm{B}} - \left[\sum_i 2\rho_i\right]\mu_{\mathrm{A}}\end{aligned}$$

which can be rearranged into

$$\mu_1 - \mu_2 = \frac{2\sum_i \rho_i}{\sum_i \left(1 - T_i + R_i\right)\rho_i}\left(\mu_{\mathrm{A}} - \mu_{\mathrm{B}}\right)$$

Substituting this into equation 43 gives the required result.

Returning to the 1st pair of equations we note that the 1st refers to the right hand lead and the 2nd to the left hand lead. Gneralising these we get

$$
\begin{aligned}
\sum_i^{N_R} T_i \rho_i^R \left(\mu_1 - \mu_{\mathrm{B}}\right) &= \sum_i^{N_R} \left(2 - T_i\right) \rho_i^R \left(\mu_{\mathrm{B}} - \mu_2\right) \\
\sum_i^{N_L} \left(1 + R_i\right) \rho_i^L \left(\mu_1 - \mu_{\mathrm{A}}\right) &= \sum_i^{N_L} \left(1 - R_i\right) \rho_i^L \left(\mu_{\mathrm{A}} - \mu_2\right) \quad .
\end{aligned}
$$

where we 1st sort the μs

$$
\begin{aligned}
2\left[\sum_i^{N_R} \rho_i^R\right] \mu_2 + \left[\sum_i^{N_R} T_i \rho_i^R\right] \left(\mu_1 - \mu_2\right) &= 2\left[\sum_i^{N_R} \rho_i^R\right] \mu_{\mathrm{B}} \\
\left[\sum_i^{N_L} \rho_i^L\right] \left(\mu_1 + \mu_2\right) + \left[\sum_i^{N_L} R_i \rho_i^L\right] \left(\mu_1 - \mu_2\right) &= 2\left[\sum_i^{N_L} \rho_i^L\right] \mu_{\mathrm{A}}
\end{aligned}
$$

Note at this point that the factors on the right hand side are simply the total density of states over all channels. Dividing each of the equations by these density of states factors and subtracting the 1st from the 2nd gives

$$
\left\{1 + \left[\sum_i^{N_L} \rho_i^L\right]^{-1} \left[\sum_i^{N_L} R_i \rho_i^L\right] - \left[\sum_i^{N_R} \rho_i^R\right]^{-1} \left[\sum_i^{N_R} T_i \rho_i^R\right]\right\} \left(\mu_1 - \mu_2\right) = 2\left(\mu_A - \mu_B\right)
$$

which leads immediately to

$$
G = \frac{4e^2}{h} \frac{\sum_i^{N_R} T_i}{1 + \left[\sum_i^{N_L} \rho_i^L\right]^{-1} \left[\sum_i^{N_L} R_i \rho_i^L\right] - \left[\sum_i^{N_R} \rho_i^R\right]^{-1} \left[\sum_i^{N_R} T_i \rho_i^R\right]}
$$

which is the required result.

8. We can write equation 48 in matrix form making use of the symmetry $T_{ij} = T_{ji}$ and of the sum rules. We also note that the current in leads 1 and 2 should be zero and in leads 3 and 4 should be opposite. This gives

$$
\begin{aligned}
0 &= (T_{12} + T_{13} + T_{14})\mu_1 - T_{12}\mu_2 - T_{13}\mu_3 - T_{14}\mu_4 \\
0 &= -T_{12}\mu_1 (T_{12} + T_{23} + T_{24})\mu_2 - T_{23}\mu_3 - T_{24}\mu_4 \\
I &= -T_{13}\mu_1 - T_{23}\mu_2 (T_{13} + T_{23} + T_{34})\mu_3 - T_{34}\mu_4 \\
-I &= -T_{14}\mu_1 - T_{24}\mu_2 - T_{34}\mu_3 (T_{14} + T_{24} + T_{34})\mu_4
\end{aligned}
$$

We don't need μ_3 and μ_4 so we will solve the 1st 2 equations for these quantities and subsitute them into the 3rd & 4th equations.

$$
\begin{aligned}
&\mu_3 + \mu_4 = (\mu_1 + \mu_2) \\
&\quad - \frac{T_{13}T_{12} + T_{23}T_{12} + T_{13}T_{23} - T_{12}T_{24} - T_{14}T_{12} - T_{14}T_{24}}{T_{13}T_{24} - T_{23}T_{14}} \left(\mu_1 - \mu_2\right) \\
&\mu_3 - \mu_4 = \\
&\quad \frac{T_{13}T_{12} + T_{13}T_{23} + T_{13}T_{24} + T_{12}T_{24} + T_{23}T_{12} + T_{14}T_{12} + T_{23}T_{14} + T_{14}T_{24}}{T_{13}T_{24} - T_{23}T_{14}} \left(\mu_1 - \mu_2\right)
\end{aligned}
$$

Substituting this into the 3rd equation (the 4th gives the same result) gives

$$I = \frac{\mu_1 - \mu_2}{T_{13}T_{24} - T_{23}T_{14}} \left[\begin{array}{r} T_{13}T_{23}T_{14} + T_{14}T_{13}T_{12} + T_{14}T_{13}T_{24} + T_{14}T_{23}T_{12} \\ +T_{12}T_{13}T_{24} + T_{23}T_{13}T_{24} + T_{23}T_{12}T_{24} + T_{23}T_{14}T_{24} \\ +T_{34}T_{12}T_{24} + T_{34}T_{14}T_{12} + T_{34}T_{14}T_{24} + T_{34}T_{13}T_{24} \\ +T_{34}T_{23}T_{14} + T_{34}T_{13}T_{12} + T_{34}T_{13}T_{23} + T_{34}T_{23}T_{12} \end{array} \right]$$

Invert this to obtain $\mu_1 - \mu_2$ and hence calculate $\mathcal{R}_{12,34}$.

9. In the case of $R_{12,34}$ the classical current flows through the central wire or loop. In addition to the classical current there will be universal conductance fluctuations, apparently random fluctuations as a function of (e.g.) magnetic field with amplitude $\Delta G \approx e^2/h$, where G is the conductance or inverse resistance. In case (b) there will be a periodic variation with a period given by the number of flux quanta which thread the loop. This flux is $\Phi = B\pi r^2$, where $r = 0.25\mu$, which is $B\pi r^2 \cdot e/h$ flux quanta. In other words the periodicity is $\Delta B = h/\pi r^2 e$. In the case of $R_{13,24}$ the classical voltage is zero. In this case we expect to observe the above fluctuations but with a mean conductance of zero.

This system is supposed to be equivalent to a macroscopic solid or to the Sharvin-Sharvin system. The technology is not able to guarantee that the rings are really similar. Each ring will give an oscillation with the field as above but with a different phase. The net effect is that all the periodic oscillations cancel. The backscattering contribution will be the same for each ring, however, and this will give a weaker oscillation with a period half of that expected.