Section 14: Solution of Partial Differential Equations; the Wave Equation

14. 1. Solution Methods

The classical methods for solving PDEs are

1. Separation of Variables—idea is to reduce a PDE of N variables to N ODEs. You have used this method extensively in last year and we will not develop it further here.
2. Integral transform and Green functions method

14. 2. Wave equation—D'Alembert's solution

First as a revision of the method of Fourier transform we consider the one-dimensional (or 1+1 including time) homogeneous wave equation.

$$u_{xx} - \frac{1}{c^2}u_{tt} = 0 \qquad -\infty < x < \infty \qquad u(x,0) = a(x) \quad u_t(x,0) = b(x)\ .$$

We already saw by the method of characteristics that the general solution is of the form $u = f(x-ct) + g(x+ct)$ and here we could fit the boundary conditions to this expression (see e.g. *Riley, Hobson, Bence* 16.4). Instead we start from scratch and use the Fourier transform. (For systems that are translationally invariant i.e. no boundary conditions at finite x the F.T. is generally the preferred method.)

Let

$$F(k,t) = \int_{-\infty}^{\infty} dx\, u(x,t)e^{-ikx}\ .$$

Transforming the wave equation gives

$$\begin{aligned} -k^2F(k,t) &= \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} \\ \Rightarrow &\qquad F(k,t) = A(k)\cos(kct) + B(k)\sin(kct) \end{aligned}$$

The boundary conditions become, with $\widetilde{a}(k) = F[a(x)]$ and $\widetilde{b}(k) = F[b(x)]$

$$F(k,0) = \widetilde{a}(k) = A(k) \qquad \frac{\partial F(k,0)}{\partial t} = \widetilde{b}(k) = kcB(k)$$

$$\Rightarrow \qquad F(k,t) = \widetilde{a}(k)\cos(kct) + \frac{\widetilde{b}(k)}{kc}\sin(kct)$$

We invoke the convolution theorem to deduce

$$u(x,t) = a(x) * F^{-1}[\cos(kct)] + b(x) * F^{-1}\left[\frac{\sin(kct)}{kc}\right]$$

Now we recall integral representation of the dirac delta and Heaviside functions to deduce

$$F^{-1}[\cos(kct)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk\,e^{ikx}}{2}\left[e^{ikct} + e^{-ikct}\right] = \frac{1}{2}\left[\delta(x+ct) + \delta(x-ct)\right]$$

$$F^{-1}\left[\frac{\sin(kct)}{kc}\right] = \frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{dk\,e^{ikx}}{ck}\left[e^{ikct} - e^{-ikct}\right] = \frac{1}{2c}\left[\theta(x+ct) - \theta(x-ct)\right]$$

Then we can carry out the convolutions to obtain

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}[a(x-ct)+a(x+ct)] + \frac{1}{2c}\int_{-\infty}^{\infty} dy\, b(x-y)\,[\theta(y+ct)-\theta(y-ct)] \\
&= \frac{1}{2}[a(x-ct)+a(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} dx' b(x')
\end{aligned}
$$

The terms in the square bracket represents the sum of the initial displacement $a(x)$ translated by ct to the left and ct to the right. The integral represents the cumulative effect of the initial velocity profile $g(x)$. To see this last point more clearly let us consider the example

$$
a(x) = 0 \qquad b(x) = \delta(x)
$$

which we can think of as a unit impulse at $x = t = 0$ e.g. a plucked guitar string. Then we obtain

$$
u(x,t) = \frac{1}{2c}[\theta(x+ct)-\theta(x-ct)] = \begin{cases} \frac{1}{2c} & \text{for} \quad -ct < x < ct \\ 0 & \text{otherwise} \end{cases}
$$

Thus the effect of the impulse at $x = 0$ propagates to the right and left with speed c creating a 'characteristic cone' or 'domain of influence' delineated by the two characteristics $x = \pm ct$. You should sketch this

Figure 1: Domain of influence for an impulse at the origin

Aside: It is interesting to note that there is nothing to stop us considering $t < 0$ then our solution gives

$$
u(x,t) = -\frac{1}{2c}[\theta(x+c|t|)-\theta(x-c|t|)] = \begin{cases} -\frac{1}{2c} & \text{for} \quad -c|t| < x < c|t| \\ 0 & \text{otherwise} \end{cases}
$$

Thus somehow we have a negative signal propagating into the past with speed c – a wave equation admits solutions propagating backwards in time!

When playing with the (classical) wave equation and guitar strings one is not usually interested in backwards time travel so $t < 0$ is not usually considered. Later we shall see how this solution has stemmed from the way we carried out the inverse Fourier transform and in particular our integral representation of the step function.

14. 3. Inhomogeneous wave equation

We now consider the inhomogeneous wave equation in three (spatial) dimensions

$$\nabla^2 u(\underline{x},t) - \frac{1}{c^2}\frac{\partial^2 u(\underline{x},t)}{\partial t^2} = f(\underline{x},t) \ .$$

We define the Green function $G(\underline{x},t;\underline{x}',t')$ as a function which satisfies

$$\nabla^2 G(\underline{x},t;\underline{x}',t') - \frac{1}{c^2}\frac{\partial^2 G(\underline{x},t;\underline{x}',t')}{\partial t^2} = \delta(\underline{x}-\underline{x}')\delta(t-t') \tag{1}$$

where the Laplacian and time derivative act on the *unprimed* variables. As usual we construct a particular solution by convolving the external influence f with the Green function:

$$u(\underline{x},t) = \int d^3x'\, dt'\, f(\underline{x}',t) G(\underline{x},t;\underline{x}',t') \ .$$

Now clearly G is a function of $\underline{x}-\underline{x}'$ and $t-t'$ therefore we may without loss of generality take $\underline{x}' = t' = 0$ in (1) and recover the general case at the end of the calculation.

Thus $G(\underline{x},t)$ is the response to a disturbance at $\underline{x}=0$, $t=0$.

We begin to solve for $G(\underline{x},t)$ by taking the Fourier transform in space and time

$$\widetilde{G}(\underline{k},\omega) = \int d^3x \int dt\, G(\underline{x},t) e^{-i(\underline{k}\cdot\underline{x}-\omega t)}$$

note the common convention that the Fourier variable ω, conjugate to t, is opposite sign to $\underline{k}$ the variable conjugate to $\underline{x}$.

Transforming (1) gives

$$\begin{aligned}\left(-k^2 + \frac{\omega^2}{c^2}\right)\widetilde{G}(\underline{k},\omega) &= 1 \\ \Rightarrow G(\underline{k},\omega) = \frac{c^2}{\omega^2 - k^2c^2}\end{aligned}$$

We are left to invert the F.T.

$$G(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\,\frac{c^2}{\omega^2-k^2c^2} e^{i(\underline{k}\cdot\underline{x}-\omega t)} \ .$$

First let us do some work on the integral over k-space

$$\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\underline{k}\cdot\underline{x}}}{\omega^2-k^2c^2}$$

We go into spherical polar co-ordinates in k-space. Since $\underline{x}$ is held constant here it is just some vector and we have the freedom to choose the '$\underline{e}_3$-axis' for our k-space spherical polars along the $\underline{x}$ direction. Then $\underline{k}\cdot\underline{x} = kr\cos\theta$ and we have

$$\begin{aligned}&\int \frac{dk\, d\theta\, d\phi}{(2\pi)^3}\frac{k^2\sin\theta}{\omega^2-k^2c^2}e^{ikr\cos\theta} \qquad \text{where} \quad r = |\underline{x}| \\ &= \frac{2\pi}{(2\pi)^3}\int_0^\infty dk\,\frac{k^2}{\omega^2-k^2c^2}\left[-\frac{e^{ikr\cos\theta}}{ikr}\right]_0^\pi = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk\,\frac{k}{ir}\frac{e^{ikr}}{\omega^2-k^2c^2}\end{aligned}$$

Thus we have performed the angular integration over θ, ϕ and we are left with

$$G(\underline{x}, t) = \frac{1}{(2\pi)^3} \frac{c^2}{ir} \int_{-\infty}^{\infty} dk\, k e^{ikr} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{(\omega - kc)(\omega + kc)} \,.$$

In the ω integral there are poles at $\omega = c|k|$ and $\omega = -c|k|$. See figure

Figure 2: Poles in complex ω plane in calculation of the Green function

For $t > 0$ we close the ω contour by a big semicircle in the lower half plane (because of $e^{-i\omega t}$)

For $t < 0$ we close the ω contour by a big semicircle in the upper half plane.

In order to proceed we have to decide how to bypass the poles on the integration contour. What to do will be specified by the boundary conditions of the problem (recall that Green functions always depend on the boundary conditions).

First note that including the poles within a closed integration contour will result in residues from the ω integral proportional to $e^{ik(r \pm ct)}$ which lead to contributions $\delta(r \pm ct)$ in G.

Now recall that we want G to be the *response* to a disturbance at $\underline{x} = 0$ $t = 0$. i.e. we want **causality** to hold and $G = 0$ for $t < 0$. This implies that the contour must bypass the poles by going above them so that when $t < 0$ they are not inside the closed contour.

However the closed contour for $t > 0$ (which is in a clockwise sense) will include the poles and we obtain by the residue theorem

$$t > 0 \qquad G(\underline{x}, t) = \frac{-2\pi i}{(2\pi)^3} \frac{c^2}{ir} \int_{-\infty}^{\infty} dk\, k e^{ikr} \left[\frac{e^{-ikct}}{2kc} - \frac{e^{ikct}}{2kc} \right] = \frac{1}{4\pi c r} \left[\delta(r + ct) - \delta(r - ct) \right]$$

Because $t > 0$ $r > 0$ the first delta becomes irrelevant and finally, putting back in $\underline{x}'$ t', we have

$$G(\underline{x} - \underline{x}', t - t') = \begin{cases} 0 & t < t' \\ -\frac{c}{4\pi|\underline{x} - \underline{x}'|}\, \delta(|\underline{x} - \underline{x}'| - c|t - t'|) & t > t' \end{cases}$$