Section 11: Eigenfunction Expansion of Green Functions

In this lecture we see how to expand a Green function in terms of eigenfunctions of the underlying Sturm-Liouville problem. First we review Hermitian matrices

11. 1. Hermitian matrices

Hermitian matrices satisfy $H_{ij} = H^*_{ji} = H^\dagger_{ij}$ where $H^\dagger$ is the Hermitian conjugate of H. You should recall that Hermitian matrices have real eigenvalues λ_n such that

$$H\,|n\rangle = \lambda_n\,|n\rangle$$

(where we use 'bra ket' notation). $|n\rangle$ is the eigenvector and (nondegenerate) eigenvectors are orthogonal.

We may write $\langle n|m\rangle = \delta_{n,m}$ (orthonormality of eigenvectors) also we have 'completeness' which means the eigenvectors span the vector space and we may write

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

which implies

$$H = H\sum_n |n\rangle\langle n| = \sum_n \lambda_n |n\rangle\langle n|\ .$$

Thus to solve the equation

$$H\,|x\rangle = |b\rangle$$

we take a scalar product

$$\begin{aligned}\langle m|\,H\,|x\rangle &= \sum_n \lambda_n \delta_{mn}\,\langle n|x\rangle = \lambda_m\,\langle m|x\rangle = \langle m|b\rangle \\ \Rightarrow |x\rangle &= \sum_n |n\rangle\,\langle n|x\rangle = \sum_n \left[\frac{|n\rangle\langle n|}{\lambda_n}\right]|b\rangle = H^{-1}\,|b\rangle\end{aligned}$$

i.e. we have an expression for H^{-1} in terms of the eigenvectors of H.

If H has an eigenvalue $\lambda_0 = 0$ then H^{-1} doesn't exist. Nevertheless we can still solve $H\,|x\rangle = |b\rangle$ in the case where $\langle 0|b\rangle = 0$. For then $\lambda_m\,\langle m|x\rangle = \langle m|b\rangle$ still holds $\forall m$ and

$$|x\rangle = \sum_{n\neq 0} \frac{|n\rangle\langle n|}{\lambda_n}\,|b\rangle + A\,|0\rangle$$

where A is an arbitrary constant.

11. 2. Hermitian Operators

We now consider the Sturm-Liouville eigenvalue problem

$$\mathcal{L}(x)u(x) = \lambda u(x)$$

with some boundary conditions imposed. An operator $\mathcal{L}$ is Hermitian if

$$\begin{aligned}\int dx\ u^*(x)\mathcal{L}(x)v(x) &= \left[\int dx\ v^*(x)\mathcal{L}(x)u(x)\right]^* \\ \text{c.f}\quad \mathcal{L}_{uv} &= \mathcal{L}^*_{vu} = \mathcal{L}^\dagger_{uv}\end{aligned}$$

In the same way as for Hermitian matrices we can show that Hermitian operators have real eigenvalues and the eigenfunctions $\phi_n(x)$ are orthogonal

$$\begin{array}{lll}\text{if} & \mathcal{L}(x)\phi_m(x) = \lambda_m\phi_m(x) & \int dx\ \phi_n^*(x)\mathcal{L}(x)\phi_m(x) = \lambda_m \int dx\ \phi_n^*(x)\phi_m(x) \\ \text{and} & \mathcal{L}(x)\phi_n(x) = \lambda_n\phi_n(x) & \int dx\ \phi_m^*(x)\mathcal{L}(x)\phi_n(x) = \lambda_n \int dx\ \phi_m^*(x)\phi_n(x)\end{array}$$

$$\begin{array}{ll}\text{since } \mathcal{L} \text{ Hermitian} & \Rightarrow (\lambda_m - \lambda_n^*) \int dx\ \phi_n^*(x)\phi_m(x) = 0 \\ & \Rightarrow \lambda_n \quad \text{are real and} \qquad \int dx\ \phi_n^*(x)\phi_m(x) = \delta_{nm}\end{array}$$

Example: $\dfrac{d^2\, u}{dx^2} = 0 \qquad \text{b.c.}\quad u(0) = u(L) = 0$

First verify that $\dfrac{d^2\, u}{dx^2}$ is an Hermitian operator:

$$\int_0^L u^* \frac{d^2 v}{dx^2} dx = \left[u^* \frac{dv}{dx}\right]_0^L - \int_0^L dx\, \frac{du^*}{dx}\frac{dv}{dx} = \left[u^*\frac{dv}{dx}\right]_0^L - \left[\frac{du^*}{dx}v\right]_0^L + \int_0^L dx\, v^* \frac{d^2 u}{dx^2}$$

and the boundary conditions ensure that the operator is Hermitian.

In this example the eigenfunctions and eigenvalues are of d^2/dx^2 obeying the b.c.s are

$$\phi_n(x) = \left(\frac{2}{L}\right)^{1/2} \sin\frac{n\pi x}{L} \qquad \lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

and form a *discrete* (i.e. countable) spectrum and that λ is bounded from above but not below. Note how the b.c.s impose the spectrum.

We can check orthogonality of the eigenfunctions and completeness:

$$\begin{aligned}\frac{2}{L}\int_0^L dx \sin\left(\frac{\pi m x}{L}\right) \sin\left(\frac{\pi n x}{L}\right) &= \delta_{mn} \quad \text{for} \quad m, n > 0 \\ \frac{2}{L}\sum_{m=1}^{\infty} \sin\left(\frac{m\pi x'}{L}\right) \sin\left(\frac{m\pi x}{L}\right) &= \delta(x - x')\end{aligned}$$

11. 3. General Sturm-Liouville problem

Consider the general Sturm Liouville problem

$$\mathcal{L}(x)\phi_n(x) = \lambda_n \rho(x)\phi_n(x) \qquad a \le x \le b$$

$$\mathcal{B}\phi_n(x) = 0 \quad \text{at} \quad x = a \quad x = b \quad e.g.\ \ \mathcal{B}(a) = \alpha + \beta\frac{d}{dx}$$

where $= \mathcal{L}(x) = \dfrac{d}{dx}\left[p(x)\dfrac{d}{dx}\right] + q(x)$. $\rho(x)$ is known as a 'weight function'.

It is easy to show that $\mathcal{L}(x)$ is Hermitian and

$$\int_a^b \rho(x)\phi_n^*(x)\phi_m(x)dx = \delta_{nm} \qquad \text{c.f.} \quad \langle\phi_n|\phi_m\rangle = \delta_{nm}$$

$$\sum_n \phi_n(x)\rho(x')\phi_n^*(x') = \delta(x-x') \qquad \text{c.f.} \quad \sum_n |\phi_n\rangle\langle\phi_n| = \delta(x-x')$$

$$f(x) = \sum_n C_n\phi_n(x) \qquad \text{c.f.} \quad |f\rangle = \sum_n C_n\,|\phi_n\rangle$$

$$C_m = \int dx\rho(x)\phi_m(x)f(x) \qquad \text{c.f.} \quad C_m = \langle\phi_m|f\rangle$$

The analogy with the bra and ket vectors is to think of the eignfunctions $\phi_n(x)$ as basis vectors for a vector space. Then we can expand an arbitrary function f in terms of them.

(**N.B.** The notation differs slightly from quantum mechanics; there $|\phi\rangle$ is the eigenvector in Hilbert space of an abstract operator and $\psi(x)$ denotes the wavefunction in position space. Here our S.L. operators are always written down in position space as are our eigenfunctions.)

Consider the Green function which satisfies

$$\mathcal{L}(x)G(x,x') = \delta(x-x') \qquad \mathcal{B}G(x,x') = 0 \quad \text{at} \quad x=a \quad x=b$$

We expand $G(x,x') = \sum_m C_m(x')\phi_m(x)$

$$\Rightarrow \qquad \sum_m \lambda_m\rho(x)\phi_m(x)C_m(x') = \delta(x-x')$$

Now multiply by $\phi_n^*(x)$ and integrate

$$\sum_m \lambda_m C_m(x')\delta_{mn} = \phi_n^*(x')$$

$$\Rightarrow \qquad G(x,x') = \sum_n \frac{\phi^*(x')\phi_n(x)}{\lambda_n}$$

This is sometimes known as the bilinear expansion of the Green function and should be compared to the expression in section 11.1 for H^{-1} We deduce that the Green function is basically the inverse of the Sturm Liouville operator.

Example: Green Function for Finite stretched string with periodic forcing

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = f(x)e^{-i\omega_0 t} \qquad \text{b.c.} \quad u(0) = u(L) = 0$$

Here u is the displacement of a stretched string and the rhs gives the external forcing which here is periodic in time.

We guess the time dependence as $u(x,t) = y(x)e^{-i\omega_0 t}$ which leads to

$$\frac{d^2y}{dx^2} + k_0^2 y = f(x) \qquad \text{where} \quad k_0^2 \equiv \frac{\omega_0^2}{c^2} \qquad \text{and the b.c.s are} \quad y(0) = y(L) = 0 \quad \diamondsuit$$

This is ODEis the Helmholtz equation and involves a Hermitian operator $\frac{d^2}{dx^2} + k_0^2$ for which the eigenfunctions of the Sturm-Liouville problem $\diamondsuit$ are

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin(n\pi x/L) \qquad \lambda_n = k_0^2 - \frac{n^2\pi^2}{L^2}$$

The Green function obeys

$$\frac{d^2G(x,x')}{dx^2} + k_0^2 G = \delta(x-x') \qquad G(0,x') = G(L,x') = 0$$

We assume a Fourier sine series solution to this equation i.e. we insert

$$G = \sum_{n=1}^{\infty} \gamma_n(x') \sin(n\pi x/L) \qquad \delta(x-x') = \sum_{n=1}^{\infty} \frac{2}{L} \sin(n\pi x'/L) \sin(n\pi x/L)$$

Then equating coefficients of $\sin n\pi x/L$ gives

$$-\frac{n^2\pi^2}{L^2}\gamma_n(x') + k^2\gamma_n(x') = \frac{2}{L}\sin(n\pi x'/L) \quad \Rightarrow \quad \gamma_n(x') = \frac{2}{L}\frac{\sin(n\pi x'/L)}{k^2 - n^2\pi^2/L^2}$$

$$\text{Thus} \qquad G(x,x') = \frac{2}{L}\sum_{n=1}^{\infty} \frac{\sin(n\pi x'/L)\ \sin(n\pi x'/L)}{k_0^2 - n^2\pi^2/L^2}$$

Also note the symmetry of the Green function $G(x,x') = G(x',x)$ and that the eigenfunction expansion fails if $k_0^2 = n^2\pi^2/L^2$. To understand this latter point, note that $\lambda_n = 0$ implies that the external forcing (k_0^2) coincides with an eigenfrequency of the unforced syste $(n\pi/L)^2$. Thus the system we will have resonance and the oscillations will grow unboundedly (if there is no dissipation). Compare with the matrix case in 11.1 where there is one eigenvalue zero but there may still be a solution if the rhs does not overlap with eigenvector 0. In the present case we may yet have a solution if $f(x)$ has zero overlap with the zero eigenfunction i.e. the spatial shape of the external forcing does not excite the resonance.

11. 4. Continuous spectrum

The boundary conditions of the problem may allow a **continuous** spectrum of eigenvalues.

Example: Helmholtz equation on infinite domain

$$\frac{d^2y}{dx^2} + k_0^2 y = f(x) \qquad y(\pm\infty) \quad \text{bounded}$$

The underlying eigenvalue problem is $\dfrac{d^2y}{dx^2} + k_0^2 y = \lambda y$ where $y(\pm\infty)$ bounded. The eigenfunctions are $y = \exp(\pm ikx)$ and eigenvalues $\lambda = k_0^2 - k^2$ with k cts.

We construct G by taking the Fourier transform of

$$\begin{aligned} \frac{d^2G(x,x')}{dx^2} + k_0^2 G(x,x') &= \delta(x-x') \\ \text{yielding} \qquad \left[-k^2 + k_0^2\right] g(k,x') &= e^{-ikx'} \\ \Rightarrow G(x,x') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k,x')e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik(x-x')}}{k_0^2 - k^2}dk \end{aligned}$$

This formula is just the generalisation of the sum, in the bilinear expansion for the discrete case, to an integral.