8.1 **An integral equation** [s] Use Laplace transforms to solve the integral equation

\[ f(y) = 1 + \int_0^y xe^{-x} f(y - x) \, dx. \]

8.2 **Asymptotic behaviour** [c] In the system of first order differential equations

\[ \dot{u} = Au + \alpha e^{i\omega_0 t}, \quad u(0) = 0, \]

for the complex \( n \) dimensional vector \( u(t) \), \( \alpha \) is a constant complex vector, \( A \) is a constant complex \( n \times n \) matrix, and \( \omega_0 \) is a real constant. By taking Laplace transforms show that

\[ u(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( sI - A \right)^{-1} \alpha e^{st} \frac{1}{s - i\omega_0} \, ds, \]

where \( c > \max\{0, \Re(\lambda_j)\} \), \( \{\lambda_j\} \) being the eigenvalues of \( A \). Hence show that if \( \Re(\lambda_j) < 0, \; j = 1, 2, \ldots n \), then

\[ u(t) = (i\omega_0 I - A)^{-1} \alpha e^{i\omega_0 t} + v(t), \]

where \( v(t) \) is a vector which vanishes exponentially as \( t \to \infty \).

8.3 **Laguerre Polynomials** [s] (i) The Laguerre polynomials may be defined as

\[ L_n(t) = e^t \frac{d^n}{dt^n} (t^n e^{-t}). \]

Show that \( \mathcal{L}[L_n(t)] = n! (s - 1)^n / s^{n+1} \). Now consider the Laplace transform of \( f_n \) satisfying the equation

\[ tf_n + (1 - t)f_n + nf_n = 0. \]

Deduce that \( f_n = L_n \) is a solution to this equation.

(ii) By taking its Laplace transform show that \( F(x, t) = e^{-xt/(1-x)} / (1 - x) \) is a generating function for the Laguerre polynomials, i.e.

\[ F(x, t) = \sum_{n=0}^{\infty} \frac{x^n}{n!} L_n(t). \]

8.4 **Riemann Zeta function** [s] The zeta function is defined for \( \Re z > 1 \) by \( \zeta(z) = \sum_{1}^{\infty} s^{-z} \).

By considering \( \int_0^{\infty} e^{-st} t^{z-1} \, dt \) show that

\[ \zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} \, dt. \]

Deduce from this the Hankel representation

\[ \zeta(z) = \frac{\Gamma(1 - z)}{2\pi i} \int_C \frac{t^{z-1}}{e^{-t} - 1} \, dt, \]

where \( C \) is the anticlockwise ‘loop’ contour around the branch cut from \(-\infty \) to \( 0 \) (see lecture 3).

[You will need to recall Euler’s reflection formula]
8.5 **Heaviside expansion theorem** [s]

If the Laplace transform $F(s)$ may be written as a ratio

$$F(s) = \frac{g(s)}{h(s)}$$

where $g(s)$ and $h(s)$ are analytic functions, $h(s)$ having simple isolated zeros at $s = s_i$ show that

$$f(t) = L^{-1} \left[ \frac{g(s)}{h(s)} \right] = \sum_i \frac{g(s_i)}{h'(s_i)} e^{s_i t}$$

8.6 **Bessel’s Equation of order 0** [s]

(i) 

$$t \ddot{f}(t) + \dot{f}(t) + tf(t) = 0$$

Let $F(s)$ be the Laplace Transform of $f(t)$. Show that

$$-(s^2 + 1) \frac{dF(s)}{ds} = sF(s)$$

Integrate this equation to obtain

$$F(s) = \frac{A}{(s^2 + 1)^{1/2}}$$

$A$ is the constant of integration. $A = 1$ gives the Bessel function $J_0$.

(ii) The inversion integral gives

$$J_0(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} ds \, e^{st} (s^2 + 1)^{-1/2}$$

Identify the singularities of $F(s)$ and discuss how to close the integration contour and whether branch cuts need be introduced.

(iii)* By using the inversion formula and expanding $F(s)$ about the singularities determine the large the asymptotic behaviour of $J_0(t)$ as $t \to \infty$ and the first correction to this behaviour.

(iv) Now develop a small $t$ expansion by expanding $F(s)$ for large $s$ then using the inversion formula. You should show

$$J_0(t) = \sum_{r=0}^{\infty} \frac{(-)^r (\frac{t}{2})^{2r}}{r! \, r!}.$$