EM 3 Section 12: The Displacement Current

In this lecture we complete the discussion of the fundamental laws of electromagnetism, and introduce electromagnetic waves for the first time.

12. 1. Continuity equation

Consider a *conserved quantity* for example electric charge—experimentally it is known that electric charge is always conserved.

We consider a volume V and the rate of change of the total charge Q in that volume. In the case where there is no creation or spontaneous loss of charge inside the volume we have

$$-\frac{\partial Q}{\partial t} = \oint_A \underline{J} \cdot \mathrm{d}\underline{S} \tag{1}$$

where the right hand side is a flux integral which expresses the total current *out* of the volume, therefore the left hand side has a negative sign.

Writing the left hand side as a volume integral over charge density ρ and the right hand side as a volume integral by virtue of the divergence theorem gives

$$-\frac{\partial}{\partial t} \int_V \rho \, \mathrm{d}V = \int_V \underline{\nabla} \cdot \underline{J} \, \mathrm{d}V$$

Since this must hold for an arbitrary volume V (however small) we deduce the differential form:

$$\boxed{\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot \underline{J}} \tag{2}$$

The divergence of the current density at any point is proportional to the rate of change of the charge density at that point.

This is **continuity equation** which is a statement of local conservation (here for charge). In fact it holds for any conserved quantity (mass, energy, electric charge, momentum, and even probability) and is one of the most general and useful equations in physics.

12. 2. The Displacement Current

Let us return to the differential form of Ampère's law

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \tag{3}$$

and take the divergence of both sides:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{B} = \mu_0 \underline{\nabla} \cdot \underline{J}$$

Now since the divergence of a curl is always zero we find

$$\underline{\nabla} \cdot \underline{J} = 0$$

This result is inconsistent with the continuity equation since generally (unless the charge distribution is static)

$$\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot \underline{J} \neq 0$$

To satisfy the continuity equation generally we need to modify Ampère's law (MIV) by the addition of a **displacement current** term to go along with $\underline{J}$ i.e. we want to have when we take the divergence of the modified MIV

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{B}) = \mu_0 \left(\underline{\nabla} \cdot \underline{J} + \frac{\partial \rho}{\partial t} \right) = 0 \tag{4}$$

Using Gauss' law MI we can replace ρ with the divergence of the electric field:

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{B}) = \mu_0 \left(\underline{\nabla} \cdot \underline{J} + \epsilon_0 \frac{\partial}{\partial t} (\underline{\nabla} \cdot \underline{E}) \right)$$

The order of the time derivative and the divergence of the electric field can be reversed, and the divergence operation removed from all terms to leave:

$$\boxed{\underline{\nabla} \times \underline{B} = \mu_0 \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right)} \tag{5}$$

This is the Ampère-Maxwell law (**MIV**) which holds for both static and time-varying charge distributions and fields.
Maxwell's stroke of genius was to include the displacement current term—often called the Maxwell correction—albeit for different reasons than we have given here! In any case we conclude that
The effect of a time-varying electric field is to produce an additional contribution to the curl of the magnetic field.

Is the displacement actually a current? Answer is not really (see next subsection) but it does, of course, have the dimensions of a current.

12. 3. Capacitor Paradox and Resolution

Consider the circuit in the figure which illustrates a parallel plate capacitor charging up

Figure 1: Capacitor paradox *(Griffiths fig 7.42)*

and current $I(t)$ flowing in the wire. If we want to compute $\underline{B}$ by taking an Amperian loop in the form of a circle around the wire (outside of the capacitor) then the surface S that we should take to compute

$$\oint \underline{B} \cdot \underline{\mathrm{d}l} = \mu_0 \int_S \underline{J} \cdot \underline{\mathrm{d}S}$$

does not appear to be well-defined e.g. taking $S = S_1$ as the surface of the disc in the plane of the loop gives $\int_{S_1} \underline{J} \cdot \underline{\mathrm{d}S} = I$; but taking $S = S_2$ as an extended surface which goes through the gap between the plates and which does not cross the wire gives $\int_{S_2} \underline{J} \cdot \underline{\mathrm{d}S} = 0$ since there is no current flowing between the plates. But really Ampère's law should hold independent of the surface bounded by the fixed loop.

If, on the other hand, we consider MIV with the Maxwell correction we *replace* the old Ampère's law in integral form by the new version

$$\int_S (\underline{\nabla} \times \underline{B}) \cdot \underline{\mathrm{d}S} = \oint \underline{B} \cdot \underline{\mathrm{d}l} = \mu_0 \int_S \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \cdot \underline{\mathrm{d}S}$$

Now we know (at least quasistatically) that between the plates of the capacitor, $\underline{E}$ is normal to the plates and $|\underline{E}| = \frac{Q}{\epsilon_0 A}$. Therefore $\epsilon_0 \frac{\partial \underline{E}}{\partial t}$ is a vector with magnitude $\dot{E} = \frac{I}{A}$. Thus inside the plates $\epsilon_0 \int_{S_2} \frac{\partial \underline{E}}{\partial t} \cdot \underline{\mathrm{d}S} = I$ and gives the same contribution as does $\int_{S_1} \underline{J} \cdot \underline{\mathrm{d}S}$ outside the plates—see tutorial sheet 6. Thus the capacitor paradox is resolved. Also we see that the diplacement current is not a real current as no current flows between the capacitor plates.

One final thing to notice about the displacement current term is that, due to the factor $\epsilon_0 \simeq 9 \times 10^{-12}\mathrm{C}^2/\mathrm{NM}^2$, it is typically *much* smaller than the current term. Thus, when there is a current flowing, the current term dominates the displacement current term.

12. 4. Maxwell's Equations

The laws of electromagnetism are summarised in four differential equations (MI-IV) known as Maxwell's equations:

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \tag{6}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \tag{7}$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \tag{8}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \tag{9}$$

MI and MII are Gauss' Laws for electric and magnetic fields
MIII is Faraday's law of induction
MIV is Ampère-Maxwell law including the displacement current

In the *electrostatic limit* Poisson's equation is obtained from MI & MIII:

$$\underline{E} = -\underline{\nabla} V \quad \text{and} \quad \nabla^2 V = -\frac{\rho}{\epsilon_0} \qquad \text{when} \qquad \frac{\partial \underline{B}}{\partial t} = 0$$

In the *magnetostatic limit* Poisson's equations for the magnetic vector potential are obtained from MII & MIV:

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad \text{and} \quad \nabla^2 \underline{A} = -\mu_0 \underline{J} \qquad \text{when} \qquad \frac{\partial \underline{E}}{\partial t} = 0$$

The continuity equation is obtained from MI & MIV:

$$\underline{\nabla} \cdot \underline{J} = -\frac{\partial \rho}{\partial t} \tag{10}$$

12. 5. Solution of Maxwell's Equations in Vacuo

In a vacuum there are no charges and currents present:

$$\rho = 0 \qquad \underline{J} = 0 \tag{11}$$

We take the curl of MIII

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\frac{\partial (\underline{\nabla} \times \underline{B})}{\partial t}$$

which inserting MIV yields

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\epsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

Similarly taking the curl of MIV leads to

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}) = -\epsilon_0 \mu_0 \frac{\partial^2 \underline{B}}{\partial t^2}$$

Now we make use of the vector identity (to be memorised) for a vector field $\underline{F}$:

$$\boxed{\underline{\nabla} \times (\underline{\nabla} \times \underline{F}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{F}) - \nabla^2 \underline{F}} \tag{12}$$

In the absence of charges MI becomes $\underline{\nabla} \cdot \underline{E} = 0$ and from MII $\underline{\nabla} \cdot \underline{B} = 0$, we are left with two **wave equations**:

$$\nabla^2 \underline{E} = \epsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2} \tag{13}$$

$$\nabla^2 \underline{B} = \epsilon_0 \mu_0 \frac{\partial^2 \underline{B}}{\partial t^2} \tag{14}$$

Thus we have *decoupled* the four (first order) Maxwell's equations for $\underline{B}$ and $\underline{E}$ in the vacuum, at the price of now having second order equations. But we know that the solution of these second order wave equations (to be revised next lecture) will be electromagnetic waves. The velocity of the electromagnetic waves is the speed of light:

$$c^2 = \frac{1}{\epsilon_0 \mu_0} = (3 \times 10^8 \text{ms}^{-1})^2 \tag{15}$$

Maxwell's equations predict that light, radio waves, X-rays etc. are all types of waves associated with oscillating electric and magnetic fields in a vacuum.

N.B. There are no charges present in a vacuum, and the waves propagate without the presence of matter!