EM 3 Section 16: Magnetic Media

16. 1. Magnetic Materials

When an external magnetic field is applied to a material it produces a **magnetization** of the atoms of the material. There are several different types of magnetization:

- Diamagnetism - the orbital angular momentum of the atomic electrons is increased slightly due to electromagnetic induction.
  This magnetization is *opposite* to the external magnetic field.
- Paramagnetism - if the atoms of a material have intrinsic magnetic moments, they align with the applied field, due to $U = -\underline{m} \cdot \underline{B}$.
  This magnetization is *parallel* to the external magnetic field.
- Ferromagnetism - in a few materials the intrinsic magnetic moments of the atoms $\underline{m}_{atom}$ *spontaneously align* due to mutual interactions of a quantum nature called 'exchange interactions'. They form *domains* with moments $\underline{m}_{atom}$ all in the same direction. This magnetization can form *permanent magnets*.

16. 2. The Magnetization Vector

In analogy with the polarization vector for dielectrics the **magnetization vector**, $\underline{M}$, is the key macroscopic field for magnetic media.

The infinitesimal magnet (equivalent to small current loop) in volume dV is given by the magnetic dipole moment per unit volume:

$$\boxed{\mathrm{d}\underline{m} = \underline{M}\mathrm{d}V} \tag{1}$$

The unit of magnetization $\underline{M}$ is Am^{-1}.

Figure 1: Magnetization loops

An array of small magnetic dipoles can be thought of as producing macroscopic current loops on the surface of the material. These currents circulate round the direction of $\underline{M}$, with a surface magnetization current density $\underline{j}_M$ (see figure).

Similarly, spatial variation of the magnetization can be expected to produce a bulk magnetization current.

To quantify these effects let us calculate the field of a magnetised object. Recall that the magnetic vector potential at $\underline{r}$ of a magnetic dipole at $\underline{r}'$ is

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{m} \times (\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3} \tag{2}$$

This generalises, when we replace $\underline{m}$ by $\underline{M}\mathrm{d}V'$ and integrate the magnetization over some volume V, to

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\underline{M} \times (\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3} \mathrm{d}V' \tag{3}$$

Now we recall that

$$\frac{(\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3} = \underline{\nabla}' \frac{1}{|\underline{r} - \underline{r}'|} \tag{4}$$

and use the product rule

$$\underline{\nabla}' \times \left(\frac{\underline{M}(\underline{r}')}{|\underline{r} - \underline{r}'|} \right) = \frac{1}{|\underline{r} - \underline{r}'|} \underline{\nabla}' \times \underline{M}(\underline{r}') + \underline{\nabla}' \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \times \underline{M}(\underline{r}')$$

to obtain

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int_V \left[\frac{1}{|\underline{r} - \underline{r}'|} \underline{\nabla}' \times \underline{M}(\underline{r}') - \underline{\nabla}' \times \left(\frac{\underline{M}}{|\underline{r} - \underline{r}'|} \right) \right] \mathrm{d}V'$$

We can rewrite the second integral as a surface integral (see tutorial sheet 9) to obtain

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int_V \frac{1}{|\underline{r} - \underline{r}'|} \underline{\nabla}' \times \underline{M}(\underline{r}') \mathrm{d}V' + \frac{\mu_0}{4\pi} \oint_S \frac{1}{|\underline{r} - \underline{r}'|} \underline{M}(\underline{r}') \times \mathrm{d}\underline{S}' \tag{5}$$

Now, the first term on the right hand side is equivalent to the potential due to a *volume* current in V

$$\boxed{\underline{J}_M = \underline{\nabla} \times \underline{M}} \tag{6}$$

and the second term is equivalent to the potential due to a *surface* current on S (normal $\underline{\hat{n}}$)

$$\boxed{\underline{j}_M = \underline{M} \times \underline{\hat{n}}} \tag{7}$$

We use the subscript M to indicate that these are effective *magnetization* currents resulting from the superposition of microscopic current loops. The volume currents (6) come from how the magnetization curls about a point—a spatial variation in the magnetization field. The surface current (7) occurs even for constant magnetization.

Example: Bar magnet "A cylindrical bar magnet has uniform magnetization M along its axis. To what current distribution is this equivalent?"

Now $\underline{M}$ is uniform so $\underline{\nabla} \times \underline{M} = 0$ and no bulk $\underline{J}_M$

Surface current density $\underline{j}_{mag} = \underline{M} \times \underline{\hat{n}} = M\underline{e}_z \times \underline{e}_\rho = M\underline{e}_\phi$ has magnitude M and is 'solenoidal', i.e. resembling a solenoid with current flowing circumferentially

Example: Toroidal magnet "A long cylindrical bar magnet of uniform $\underline{M}$ is bent into a loop. What is the equivalent current distribution?"

Curl in cylindrical polars (ρ, ϕ, z) reads:

$$\begin{aligned} \underline{\nabla} \times \underline{M} &= \left[\frac{1}{\rho}\frac{\partial M_z}{\partial \phi} - \frac{\partial M_\phi}{\partial z}\right]\underline{e}_\rho + \left[\frac{\partial M_\rho}{\partial z} - \frac{\partial M_z}{\partial \rho}\right]\underline{e}_\phi \\ &+ \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho M_\phi) - \frac{\partial M_\rho}{\partial \phi}\right]\underline{e}_z \end{aligned}$$

Direction of $\underline{M}$ is circumferential $\underline{M} = M\underline{e}_\phi$. In the curl formula, the only survivor is

$$\underline{\nabla} \times \underline{M} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho M)\underline{e}_z = \frac{M}{\rho}\underline{e}_z$$

Alongside the solenoidal (circumferential around the toroid) $j_{mag} = M$ on surface, we now have bulk magnetization current N.B. The surface current $\underline{j}_{mag} = \underline{M} \times \underline{\hat{n}}$ has constant

Figure 2: Magnetization currents in bar magnet and toroidal magnet

magnitude: larger net current on outer than inner surface. The bulk current $\underline{J}_M$ makes up the difference

16. 3. Modification to Ampere's Law

The Ampère-Maxwell law still holds for the **full** current density $\underline{J}$

$$\underline{\nabla} \times \underline{B} = \mu_0 \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t}\right)$$

The key idea is to divide this into three contributions $\underline{J} = \underline{J}_f + \underline{J}_M + \underline{J}_P$

$\underline{J}_f$, current of free charges i.e. the *conduction* current

$\underline{J}_M = \underline{\nabla} \times \underline{M}$, *magnetization* current we have just met

$\underline{J}_P =$ *polarization* current — this new term comes from electric dipoles moving around

To find $\underline{J}_P$ we use the (definition) $\rho_P = -\underline{\nabla} \cdot \underline{P}$ and the continuity equation

$$\dot{\rho}_P = -\underline{\nabla} \cdot \underline{J}_P$$

from whicn we deduce

$$\boxed{\underline{J}_P = \frac{\partial \underline{P}}{\partial t}} \qquad (8)$$

We would like Ampère-Maxwell in terms of $\underline{J}_f$ only:

$$\begin{aligned}
\underline{\nabla} \times \underline{B} &= \mu_0 \left(\underline{J}_f + \underline{J}_M + \underline{J}_P + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \\
&= \mu_0 \left(\underline{J}_f + \underline{\nabla} \times \underline{M} + \frac{\partial \underline{P}}{\partial t} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \\
&= \mu_0 \left(\underline{J}_f + \underline{\nabla} \times \underline{M} + \frac{\partial \underline{D}}{\partial t} \right)
\end{aligned}$$

where we have used the definition of $\underline{D}$. Now shift $\underline{\nabla} \times \underline{M}$ onto left, divide by μ_0:

$$\underline{\nabla} \times \left(\frac{\underline{B}}{\mu_0} - \underline{M} \right) = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}$$

We define

$$\boxed{\underline{H} = \frac{\underline{B}}{\mu_0} - \underline{M}} \qquad (9)$$

then

$$\boxed{\underline{\nabla} \times \underline{H} = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}} \qquad (10)$$

which is **Ampère-Maxwell law in media**. The Integral form of Ampère-Maxwell reads

$$\oint_C \underline{H} \cdot \underline{\mathrm{d}l} = \int_S (\underline{J}_f + \partial \underline{D}/\partial t) \cdot \underline{\mathrm{d}S}$$

where C is a closed circuit bounding S

We run into difficulties in terminology for $\underline{B}$, $\underline{H}$. It is actually simplest and easiest to call them 'magnetic field B' (units Tesla) and 'magnetic field H' in units of Am^{-1}. But be warned in some texts $\underline{B}$ is the 'magnetic field' and $\underline{H}$ is the 'auxiliary field'; in others $\underline{B}$ is the 'magnetic flux density' and $\underline{H}$ is the 'magnetic field strength' (which is really confusing!)

Since MII and MIII do not need to be modified as they contain no $\underline{J}$ or ρ, we now basically have all the macroscopic Maxwell's equations which hold in media. See next lecture for summary

The **magnetic susceptibility** χ_M describes the relationship between magnetization and applied field, by relating $\underline{M}$ to $\underline{H}$. We will assume again an *LIH medium* (linear, isotropic, homogeneous). Then the relation may be written

$$\underline{M} = \chi_M \underline{H} \qquad (11)$$

Warning—some books, e.g. Grant & Phillips, use $\chi_B \underline{B} = \mu_0 \underline{M}$

The equivalent of the dielectric constant is known as the **relative permeability** of a material, μ_r:

$$\boxed{\underline{B} = \mu_r \mu_0 \underline{H}} \qquad (12)$$

where $(\mu_r - 1) = \chi_M$. The limit of no magnetization is $\chi_M = 0$ and $\mu_r = 1$.

In contrast to dielectrics, the magnetic susceptibility χ_M can be either positive or negative, and $\mu_r < 1$ or $\mu_r > 1$.