16. 1. Magnetic Materials

When an external magnetic field is applied to a material it produces a **magnetization** of the atoms of the material. There are several different types of magnetization:

- **Diamagnetism** - the orbital angular momentum of the atomic electrons is increased slightly due to electromagnetic induction. This magnetization is *opposite* to the external magnetic field.

- **Paramagnetism** - if the atoms of a material have intrinsic magnetic moments, they align with the applied field, due to $U = -m \cdot B$. This magnetization is *parallel* to the external magnetic field.

- **Ferromagnetism** - in a few materials the intrinsic magnetic moments of the atoms $m_{\text{atom}}$ *spontaneously align* due to mutual interactions of a quantum nature called ‘exchange interactions’. They form *domains* with moments $m_{\text{atom}}$ all in the same direction. This magnetization can form *permanent magnets*.

16. 2. The Magnetization Vector

In analogy with the polarization vector for dielectrics the **magnetization vector**, $M$, is the key macroscopic field for magnetic media.

The infinitesimal magnet (equivalent to small current loop) in volume $dV$ is given by the magnetic dipole moment per unit volume:

$$dm = M dV \quad (1)$$

The unit of magnetization $M$ is Am$^{-1}$.

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**Figure 1: Magnetization loops**

An array of small magnetic dipoles can be thought of as producing macroscopic current loops on the surface of the material. These currents circulate round the direction of $M$, with a surface magnetization current density $\hat{j}_M$ (see figure).
Similarly, spatial variation of the magnetization can be expected to produce a bulk magnetization current.

To quantify these effects let us calculate the field of a magnetised object. Recall that the magnetic vector potential at \( r \) of a magnetic dipole at \( r' \) is

\[
A(r) = \frac{\mu_0 m \times (r - r')}{|r - r'|^3}
\]

(2)

This generalises, when we replace \( m \) by \( M dV' \) and integrate the magnetization over some volume \( V \), to

\[
A(r) = \frac{\mu_0}{4\pi} \int_V \frac{M \times (r - r')}{|r - r'|^3} \, dV'
\]

(3)

Now we recall that

\[
\frac{(r - r')}{|r - r'|^3} = \nabla' \frac{1}{|r - r'|}
\]

(4)

and use the product rule

\[
\nabla' \times \left( \frac{M(r')}{|r - r'|} \right) = \frac{1}{|r - r'|} \nabla' \times M(r') + \nabla' \left( \frac{1}{|r - r'|} \right) \times M(r')
\]

to obtain

\[
A(r) = \frac{\mu_0}{4\pi} \int_V \left[ \frac{1}{|r - r'|} \nabla' \times M(r') - \nabla' \left( \frac{M(r')}{|r - r'|} \right) \right] \, dV'
\]

We can rewrite the second integral as a surface integral (see tutorial sheet 9) to obtain

\[
A(r) = \frac{\mu_0}{4\pi} \int_V \frac{1}{|r - r'|} \nabla' \times M(r') \, dV' + \frac{\mu_0}{4\pi} \oint_S \frac{1}{|r - r'|} M(r') \times dS'
\]

(5)

Now, the first term on the right hand side is equivalent to the potential due to a \textit{volume} current in \( V \)

\[
J_M = \nabla \times M
\]

(6)

and the second term is equivalent to the potential due to a \textit{surface} current on \( S \) (normal \( \hat{n} \))

\[
J_M = M \times \hat{n}
\]

(7)

We use the subscript \( M \) to indicate that these are effective \textit{magnetization} currents resulting from the superposition of microscopic current loops. The volume currents (6) come from how the magnetization curls about a point—a spatial variation in the magnetization field. The surface current (7) occurs even for constant magnetization.

**Example: Bar magnet** “A cylindrical bar magnet has uniform magnetization \( M \) along its axis. To what current distribution is this equivalent?”

Now \( M \) is uniform so \( \nabla \times M = 0 \) and no bulk \( J_M \)

Surface current density \( j_{mag} = M \times \hat{n} = M \varepsilon_z \times \varepsilon_\rho = M \varepsilon_\phi \) has magnitude \( M \) and is ‘solenoidal’, i.e. resembling a solenoid with current flowing circumferentially
Example: Toroidal magnet “A long cylindrical bar magnet of uniform \( \mathbf{M} \) is bent into a loop. What is the equivalent current distribution?”

Curl in cylindrical polars \((\rho, \phi, z)\) reads:

\[
\nabla \times \mathbf{M} = \left[ \frac{1}{\rho} \frac{\partial M_z}{\partial \phi} - \frac{\partial M_\phi}{\partial z} \right] \mathbf{\hat{\rho}} + \left[ \frac{\partial M_\phi}{\partial z} - \frac{\partial M_z}{\partial \rho} \right] \mathbf{\hat{\phi}} \\
+ \frac{1}{\rho} \left[ \frac{\partial}{\partial \rho} (\rho M_\phi) - \frac{\partial M_\rho}{\partial \phi} \right] \mathbf{\hat{z}}
\]

Direction of \( \mathbf{M} \) is circumferential \( M = M \mathbf{\hat{\phi}} \). In the curl formula, the only survivor is

\[
\nabla \times \mathbf{M} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho M) \mathbf{\hat{z}} = \frac{M}{\rho} \mathbf{\hat{z}}
\]

Alongside the solenoidal (circumferential around the toroid) \( j_{\text{mag}} = M \) on surface, we now have bulk magnetization current N.B. The surface current \( j_{\text{mag}} = \mathbf{M} \times \mathbf{\hat{n}} \) has constant magnitude: larger net current on outer than inner surface. The bulk current \( J_M \) makes up the difference.

16. 3. Modification to Ampere’s Law

The Ampère-Maxwell law still holds for the full current density \( \mathbf{J} \)

\[
\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)
\]

The key idea is to divide this into three contributions \( \mathbf{J} = \mathbf{J}_f + \mathbf{J}_M + \mathbf{J}_P \)

\( \mathbf{J}_f \), current of free charges i.e. the conduction current

\( \mathbf{J}_M = \nabla \times \mathbf{M} \), magnetization current we have just met

\( \mathbf{J}_P = \text{polarization current} \) — this new term comes from electric dipoles moving around

To find \( \mathbf{J}_P \) we use the (definition) \( \rho_P = -\nabla \cdot \mathbf{P} \) and the continuity equation

\[
\dot{\rho}_P = -\nabla \cdot \mathbf{J}_P
\]
from which we deduce

$$J_P = \frac{\partial P}{\partial t}$$  \hspace{1cm} (8)

We would like Ampère-Maxwell in terms of $J_f$ only:

$$\nabla \times B = \mu_0 \left( J_f + J_M + J_P + \epsilon_0 \frac{\partial E}{\partial t} \right)$$

$$= \mu_0 \left( J_f + \nabla \times M + \frac{\partial P}{\partial t} + \epsilon_0 \frac{\partial E}{\partial t} \right)$$

$$= \mu_0 \left( J_f + \nabla \times M + \frac{\partial D}{\partial t} \right)$$

where we have used the definition of $D$. Now shift $\nabla \times M$ onto left, divide by $\mu_0$:

$$\nabla \times \left( \frac{B}{\mu_0} - M \right) = J_f + \frac{\partial D}{\partial t}$$

We define

$$H = \frac{B}{\mu_0} - M$$  \hspace{1cm} (9)

then

$$\nabla \times H = J_f + \frac{\partial D}{\partial t}$$  \hspace{1cm} (10)

which is **Ampère-Maxwell law in media**. The Integral form of Ampère-Maxwell reads

$$\oint_C H \cdot dl = \int_S (J_f + \partial D/\partial t) \cdot dS$$

where $C$ is a closed circuit bounding $S$

We run into difficulties in terminology for $B$, $H$. It is actually simplest and easiest to call them ‘magnetic field B’ (units Tesla) and ‘magnetic field H’ in units of Am$^{-1}$. But be warned in some texts $B$ is the ‘magnetic field’ and $H$ is the ‘auxiliary field’; in others $B$ is the ‘magnetic flux density’ and $H$ is the ‘magnetic field strength’ (which is really confusing!)

Since MII and MIII do not need to be modified as they contain no $J$ or $\rho$, we now basically have all the macroscopic Maxwell’s equations which hold in media. See next lecture for summary

The **magnetic susceptibility** $\chi_M$ describes the relationship between magnetization and applied field, by relating $M$ to $H$. We will assume again an $LIH$ medium (linear, isotropic, homogeneous). Then the relation may be written

$$M = \chi_M H$$  \hspace{1cm} (11)

**Warning**—some books, e.g. Grant & Phillips, use $\chi_B B = \mu_0 M$.

The equivalent of the dielectric constant is known as the **relative permeability** of a material, $\mu_r$:

$$B = \mu_r \mu_0 H$$  \hspace{1cm} (12)

where $(\mu_r - 1) = \chi_M$. The limit of no magnetization is $\chi_M = 0$ and $\mu_r = 1$.

In contrast to dielectrics, the magnetic susceptibility $\chi_M$ can be either positive or negative, and $\mu_r < 1$ or $\mu_r > 1$. 